Transformations of Cramér-Rao Bounds

ECE 6279: Spatial Array Processing

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Lecture 25

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Transformation of ML Estimates

- **Suppose we find a maximum likelihood estimate**

$$\hat{\gamma}_{ML} = \arg\max_{\gamma} \ln p(y;\gamma)$$

- **But what we really want is to find a ML estimate in terms of** $\phi = g(\gamma)$ **(assume *f* is continuous & invertible)**
- **It turns out ML estimation "commutes:"** $\hat{\phi}_{ML} = g(\hat{\gamma}_{ML})$

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Properties May Not Commute

$$\hat{\phi} = g(\hat{\gamma})$$

- **If $\hat{\gamma}$ is unbiased/efficient, that doesn't necessarily imply that $\hat{\phi}$ is unbiased/efficient**
- **If using ML estimators, $\hat{\phi}$ will be at least be *asymptotically* unbiased/ efficient**

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Transformations of Fisher Information

$$F_\phi(\phi) = \left[\frac{dg(\gamma)}{d\gamma}\right]^{-2} F_\gamma(\gamma)\Bigg|_{\gamma = g^{-1}(\phi)}$$

$$= \left[\frac{dg^{-1}(\phi)}{d\phi}\right]^{2} F_\gamma(\gamma)\Bigg|_{\gamma = g^{-1}(\phi)}$$

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Transformations of CR Bounds

$$CRB_\phi(\phi) = \left[\frac{dg(\gamma)}{d\gamma}\right]^2 CRB_\gamma(\gamma)\Bigg|_{\gamma=g^{-1}(\phi)}$$

$$= \left[\frac{dg^{-1}(\phi)}{d\phi}\right]^{-2} CRB_\gamma(\gamma)\Bigg|_{\gamma=g^{-1}(\phi)}$$

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Our Example Transformation

$$\gamma = g^{-1}(\phi) = \frac{2\pi}{\lambda} d \sin(\phi)$$

$$\phi = g(\gamma) = \sin^{-1}\left(\gamma \frac{\lambda}{2\pi d}\right)$$

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The First Way (1)

$$\frac{dg(\gamma)}{d\gamma} = \frac{d}{d\gamma}\left\{\sin^{-1}\left(\gamma\frac{\lambda}{2\pi d}\right)\right\}$$

$$= \frac{\lambda}{2\pi d}\frac{1}{\sqrt{1-\left[\gamma\frac{\lambda}{2\pi d}\right]^2}}$$

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The First Way (2)

$$\left.\frac{dg(\gamma)}{d\gamma}\right|_{\gamma=f^{-1}(\phi)} = \frac{\lambda}{2\pi d}\left.\frac{1}{\sqrt{1-\left[\gamma\frac{\lambda}{2\pi d}\right]^2}}\right|_{\gamma=\frac{2\pi}{\lambda}d\sin(\phi)}$$

$$= \frac{\lambda}{2\pi d}\frac{1}{\sqrt{1-\sin^2(\phi)}} = \frac{\lambda}{2\pi d}\frac{1}{\cos(\phi)}$$

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The Second Way

$$\frac{dg^{-1}(\phi)}{d\phi} = \frac{d}{d\phi}\left\{\frac{2\pi}{\lambda} d\sin(\phi)\right\}$$

$$= \frac{2\pi}{\lambda} d\cos(\phi)$$

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Hopefully Get the Same Answer

$$\left[\frac{dg(\gamma)}{d\gamma}\right]^2\Bigg|_{\gamma=f^{-1}(\phi)} = \left(\frac{\lambda}{2\pi d}\right)^2 \frac{1}{\cos^2(\phi)}$$

$$\left[\frac{dg^{-1}(\phi)}{d\phi}\right]^{-2} = \left[\frac{2\pi}{\lambda} d\cos(\phi)\right]^{-2} = \left(\frac{\lambda}{2\pi d}\right)^2 \frac{1}{\cos^2(\phi)}$$

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Putting it All Together

From last lecture: $$\mathrm{var}_\gamma[\hat{\gamma}(\underline{y})] \gtrsim \frac{6}{M^3 L(SNR)}$$

$$\mathrm{var}_\phi[\hat{\phi}(\underline{y})] \gtrsim \frac{6}{M^3 L(SNR)} \left(\frac{\lambda}{2\pi d}\right)^2 \frac{1}{\cos^2(\phi)}$$

$$\mathrm{var}_0[\hat{\phi}(\underline{y})] \gtrsim \left(\frac{\lambda}{2\pi d}\right)^2 \frac{6}{M^3 L(SNR)} \qquad \mathrm{var}_{\pi/2}[\hat{\phi}(\underline{y})] \gtrsim \infty$$

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One Random Tidbit for This Example

From (5.9) of Stoica & Nehorai:

$$\frac{\mathrm{var}_\gamma[\hat{\gamma}_{ML}(\underline{y})]}{CRB(\gamma)} = 1 + \frac{1}{M(SNR)}$$

$$SNR = \frac{1}{L}\sum_{l=0}^{L-1} |s(l)|^2 \Big/ \sigma^2$$

$\hat{\gamma}_{ML}$ **is inefficient for finite *M*, even if** $L \to \infty$

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Defining a Gradient

$$\nabla_{\boldsymbol{\xi}}\{g(\boldsymbol{\xi})\} = \begin{bmatrix} \frac{\partial g_1(\boldsymbol{\xi})}{\partial \xi_1} & \cdots & \frac{\partial g_N(\boldsymbol{\xi})}{\partial \xi_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_1(\boldsymbol{\xi})}{\partial \xi_N} & \cdots & \frac{\partial g_N(\boldsymbol{\xi})}{\partial \xi_N} \end{bmatrix}$$

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Multivariate Transformations

- **Suppose $\alpha = g(\xi)$, continuous and invertible**
- **Nice result from p. 230 of Scharf:**

$$F_\alpha(\alpha) = [\nabla_\xi\{g(\xi)\}]^{-1} F_\xi(\xi) [\nabla_\xi^T\{g(\xi)\}]^{-1}\Big|_{\xi = g^{-1}(\alpha)}$$

$$\mathrm{cov}_\alpha\{\hat{\alpha}(\underline{y})\} \geq \nabla_\xi^T\{g(\xi)\} F_\xi^{-1}(\xi) [\nabla_\xi\{g(\xi)\}]\Big|_{\xi = g^{-1}(\alpha)}$$

- **Another version:**

$$F_\alpha(\alpha) = \nabla_\alpha\{g^{-1}(\alpha)\} F_\xi(\xi) \nabla_\alpha^T\{g^{-1}(\alpha)\}\Big|_{\xi = g^{-1}(\alpha)}$$

$$\mathrm{cov}_\alpha\{\hat{\alpha}(\underline{y})\} \geq [\nabla_\alpha^T\{g^{-1}(\alpha)\}]^{-1} F_\xi^{-1}(\xi) [\nabla_\alpha\{g^{-1}(\alpha)\}]^{-1}\Big|_{\xi = g^{-1}(\alpha)}$$

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