Introduction to Cramér-Rao Bounds

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Prof. Aaron D. Lanterman

School of Electrical & Computer Engineering
Georgia Institute of Technology
AL: 404-385-2548
<lanterma@ece.gatech.edu>

Basic Univariate Cramér-Rao Bound

- **For any unbiased estimator $\hat{\xi}$**

$$MSE = \mathrm{var}_{\xi}[\hat{\xi}(\underline{y})] \geq 1 / F(\xi)$$

where the Fisher Information is

$$F(\xi) = E_{\xi}\left\{ \left[\frac{d}{d\tilde{\xi}} \ln p(\underline{y};\tilde{\xi}) \right]^2 \Bigg|_{\tilde{\xi}=\xi} \right\}$$

Two Ways to Compute the F.I.

- **Under some common conditions,**

$$F(\xi) = E_{\xi}\left\{ \left[\frac{d}{d\tilde{\xi}} \ln p(\underline{y};\tilde{\xi}) \right]^2 \Bigg|_{\tilde{\xi}=\xi} \right\}$$

$$= -E_{\xi}\left\{ \frac{d^2}{d\tilde{\xi}^2} \ln p(\underline{y};\tilde{\xi}) \Bigg|_{\tilde{\xi}=\xi} \right\}$$

Gaussian Example, 1st Derivative (1)

- **Consider F.I. for one data point for**

$$\underline{y} \sim N(f(\xi), \sigma^2)$$

$$\frac{d}{d\xi} \ln p(y;\xi) =$$

$$\frac{d}{d\xi} \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{[y - f(\xi)]^2}{2\sigma^2} \right\} \right)$$

Gaussian Example, 1st Derivative (2)

$$\frac{d}{d\xi}\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{[y-f(\xi)]^2}{2\sigma^2}\right\}\right)$$

$$=\frac{d}{d\xi}\left\{-\frac{[y-f(\xi)]^2}{2\sigma^2}\right\}$$

$$=\frac{[y-f(\xi)]}{\sigma^2}\frac{df(\xi)}{d\xi}$$

Gaussian Example, 2st Derivative (1)

$$\frac{d^2}{d\xi^2} \ln p(y;\xi)$$

$$= \frac{d}{d\xi} \left\{ \frac{[y - f(\xi)]}{\sigma^2} \frac{df(\xi)}{d\xi} \right\}$$

$$= \frac{1}{\sigma^2} \left\{ [y - f(\xi)] \frac{d^2 f(\xi)}{d\xi^2} - \left[\frac{df(\xi)}{d\xi} \right]^2 \right\}$$

Using Derivative-Squared Version (1)

$$F(\xi) = E_{\xi}\left\{\left[\frac{\partial}{\partial \xi}\ln p(\underline{y};\xi)\right]^2\right\}$$

$$= E_{\xi}\left\{\left[\frac{[\underline{y} - f(\xi)]}{\sigma^2}\frac{df(\xi)}{d\xi}\right]^2\right\}$$

Using Derivative-Squared Version (2)

$$F(\xi) = \left[\frac{df(\xi)}{d\xi}\right]^2 \frac{E_\xi\{[\underline{y} - f(\xi)]^2\}}{(\sigma^2)^2}$$

$$= \left[\frac{df(\xi)}{d\xi}\right]^2 \Big/ \sigma^2$$

Using Double-Derivative Version (1)

$$F(\xi) = -E_{\xi}\left\{\frac{\partial^2}{\partial \xi^2} \ln p(\underline{y};\xi)\right\}$$

$$= -\frac{1}{\sigma^2} E_{\xi}\left\{[\underline{y} - f(\xi)]\frac{d^2 f(\xi)}{d\xi^2} - \left[\frac{df(\xi)}{d\xi}\right]^2\right\}$$

Using Double-Derivative Version (2)

$$-\frac{1}{\sigma^2}\left\{E_\xi[\underline{y}-f(\xi)]\frac{d^2 f(\xi)}{d\xi^2}-\left[\frac{df(\xi)}{d\xi}\right]^2\right\}$$

$$=\left[\frac{df(\xi)}{d\xi}\right]^2 \Big/ \sigma^2$$

Independent, Identically Dist. Data

- **If you have *L* i.i.d. data points, the F.I. is just L times the F.I. for one data point:**

$$F(\xi) = LF_1(\xi)$$

- **For our ex.,** $F(\xi) = L\left[\frac{df(\xi)}{d\xi}\right]^2 \Big/ \sigma^2$

$$\mathrm{var}_\xi[\hat{\xi}(\underline{y})] \geq \sigma^2 \Big/ \left\{ L\left[\frac{df(\xi)}{d\xi}\right]^2 \right\}$$

Basic Multivariate Cramér-Rao Bound

- **For any unbiased estimator $\hat{\xi}$**

$$\text{cov}_{\xi}[\hat{\xi}(\underline{y})] \geq \mathbf{F}^{-1}(\hat{\xi})$$

(A≥B means A-B is nonneg. def.)

where the entries of
of the F.I. matrix are

$$\mathbf{F}_{rc} = E_{\xi}\left\{\left[\frac{\partial}{\partial \xi_r}\ln p(\underline{y};\xi)\right]\left[\frac{\partial}{\partial \xi_c}\ln p(\underline{y};\xi)\right]\right\}$$

Nonnegative Definiteness

- **If A is nonnegative definite, then**
  - $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$ for any real vector **x**
  - Eigenvalues of **A** are nonnegative
- **Useful consequences: if A≥B,**
  - Diagonals dominated: $\mathbf{A}_{ii} \geq \mathbf{B}_{ii}$
  - Does not mean $\mathbf{A}_{rc} \geq \mathbf{B}_{rc}$ in general!
  - Total sum property:

$$\sum_{r,c} \mathbf{A}_{rc} \geq \sum_{r,c} \mathbf{B}_{rc}$$

Two Ways to Compute the F.I.M.

- **Under some common conditions:**

$$\mathbf{F}_{rc} = E_{\boldsymbol{\xi}}\left\{\left[\frac{\partial}{\partial\tilde{\xi}_r}\ln p(\underline{y};\tilde{\boldsymbol{\xi}})\right]\left[\frac{\partial}{\partial\tilde{\xi}_c}\ln p(\underline{y};\tilde{\boldsymbol{\xi}})\right]\right\}$$

$$= -E_{\boldsymbol{\xi}}\left\{\frac{\partial^2}{\partial\xi_r\partial\xi_c}\ln p(\underline{y};\boldsymbol{\xi})\right\}$$

Efficiency

- An unbiased estimator that achieves the CR bound with equality is called efficient
- If an efficient estimator exists, the maximum-likelihood estimator is it!
- Even if an estimator isn't efficient, it may be asymptotically efficient
- ML estimators are asymptotically efficient

CR Bounds for Biased Estimators

- Both univariate and multivariate versions for biased estimators exist…
- …but they require taking derivatives of the bias…
- …which requires you have an analytic form for the bias…
- …which you almost never have…
- …so it's rarely used; people usually just compute the CR bound for the unbiased estimator and then handwave