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1

$$w(\theta|y) = \frac{p_\theta(y)\,w(\theta)}{\int_{-\infty}^{\infty} p_\theta(y)\,w(\theta)\,d\theta}$$

$$\int_{-\infty}^{\infty} p_\theta(y)\,w(\theta)\,d\theta = \int_1^e \frac{1}{2} e^{-\theta|y|}\,d\theta = -\frac{1}{2|y|} e^{-\theta|y|}\Big]_1^e = \frac{1}{2|y|}\left(e^{-|y|} - e^{-e|y|}\right)$$

$$1 \leq \theta \leq e \Rightarrow w(\theta|y) = \frac{\frac{1}{2} e^{-\theta|y|}}{\frac{1}{2|y|}\left(e^{-|y|} - e^{-e|y|}\right)} = \frac{|y|\,e^{-\theta|y|}}{e^{-|y|} - e^{-e|y|}} \qquad \text{(See Fig. 1.)}$$

$$\theta < 1 \text{ or } \theta > e \Rightarrow w(\theta|y) = 0$$

(a) MAP Estimator:

$$\hat{\theta}_{MAP}(y) = \arg\max_\theta \{w(\theta|y)\} = 1$$

(b) MMSE Estimator:

Let $A \triangleq \dfrac{|y|}{e^{-|y|} - e^{-e|y|}}$

$$\hat{\theta}_{MMSE}(y) = E\{\Theta | Y = y\} = A\int_1^e \theta e^{-\theta|y|}\,d\theta = -\frac{A}{|y|}\int_1^e \theta\, de^{-\theta|y|}$$

$$= -\frac{A}{|y|}\theta e^{-\theta|y|}\Big]_1^e + \frac{A}{|y|}\int_1^e e^{-\theta|y|}\,d\theta$$

$$= \frac{A}{|y|}\left(e^{-|y|} - e\,e^{-e|y|}\right) - \frac{A}{|y|^2}\left[e^{-\theta|y|}\right]_1^e$$

$$= \frac{e^{-|y|} - e^{-e|y|+1}}{e^{-|y|} - e^{-e|y|}} + \frac{1}{|y|}$$

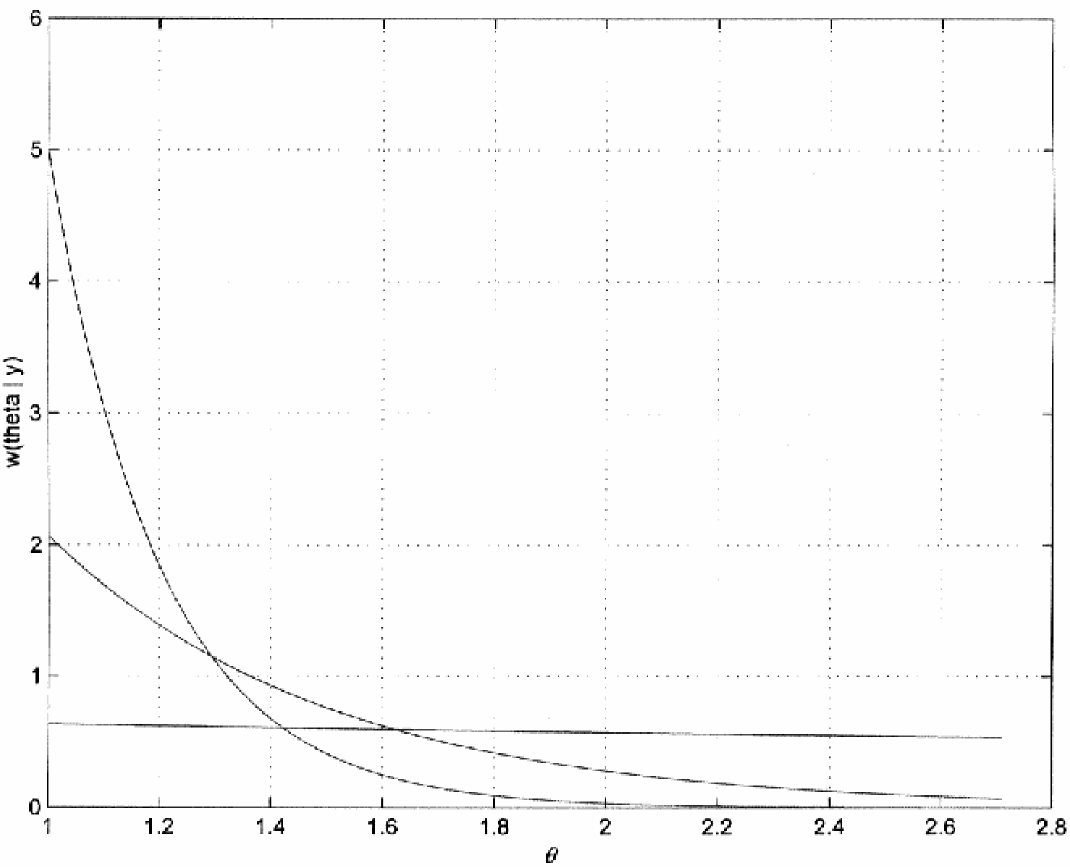

Fig 1. Exercise 1

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2. $p_\theta(y) = p(y|\theta) = p(y|\theta, S=+1)\, p(S=+1) + p(y|\theta, S=-1)\, p(S=-1)$

$$= \frac{1}{2\sqrt{2\pi}} \left\{ e^{-(y-\theta)^2/2} + e^{-(y+\theta)^2/2} \right\}$$

$$= \frac{1}{2\sqrt{2\pi}} e^{-y^2/2} e^{-\theta^2/2} \left(e^{y\theta} + e^{-y\theta} \right) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} e^{-\theta^2/2} \cosh(y\theta)$$

$$p(y) = \int_{-\infty}^{\infty} p_\theta(y)\, w(\theta)\, d\theta = \frac{K}{\sqrt{2\pi}} e^{-y^2/2} \int_0^1 \cosh(y\theta)\, d\theta$$

$$= \frac{K}{\sqrt{2\pi}} e^{-y^2/2} \left[\frac{1}{y} \sinh(y\theta) \right]_0^1 = \frac{K}{y\sqrt{2\pi}} e^{-y^2/2} \sinh(y)$$

$$w(\theta|y) = \frac{p_\theta(y)\, w(\theta)}{p(y)} = \frac{\frac{1}{\sqrt{2\pi}} e^{-y^2/2} e^{-\theta^2/2} \cosh(y\theta)\; K e^{\theta^2/2}}{\frac{K}{y\sqrt{2\pi}} e^{-y^2/2} \sinh(y)}, \quad 0 \le \theta \le 1$$

$w(\theta|y) = 0, \quad \theta \notin [0,1]$

$$\therefore\ w(\theta|y) = \begin{cases} \frac{y}{\sinh(y)} \cosh(y\theta), & 0 \le \theta \le 1 \\ 0, & \text{oth.} \end{cases}$$

(a) MMSE Estimator:

$$\hat{\theta}_{MMSE}(y) = E\{\theta | Y=y\} = \int_{-\infty}^{\infty} \theta\, w(\theta|y)\, d\theta = \frac{y}{\sinh(y)} \int_0^1 \theta \cosh(y\theta)\, d\theta$$

$$= \frac{1}{\sinh(y)} \int_0^1 \theta\, d \sinh(y\theta)$$

$$= \frac{1}{\sinh(y)} \left[\theta \sinh(y\theta) \right]_0^1 - \frac{1}{\sinh(y)} \int_0^1 \sinh(y\theta)\, d\theta$$

$$= 1 - \frac{1}{\sinh(y)} \left[\frac{1}{y} \cosh(y\theta) \right]_0^1 = 1 - \frac{1}{y \sinh(y)} \left[\cosh(y) - 1 \right]$$

 (See Fig. 2)

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(b) MAP Estimator:

$$\hat{\theta}_{MAP}(y) = \arg\max_{\theta} \left\{ w(\theta|y) \right\} = \arg\max_{0 \le \theta \le 1} \left\{ \cosh(y\theta) \right\} = 1$$

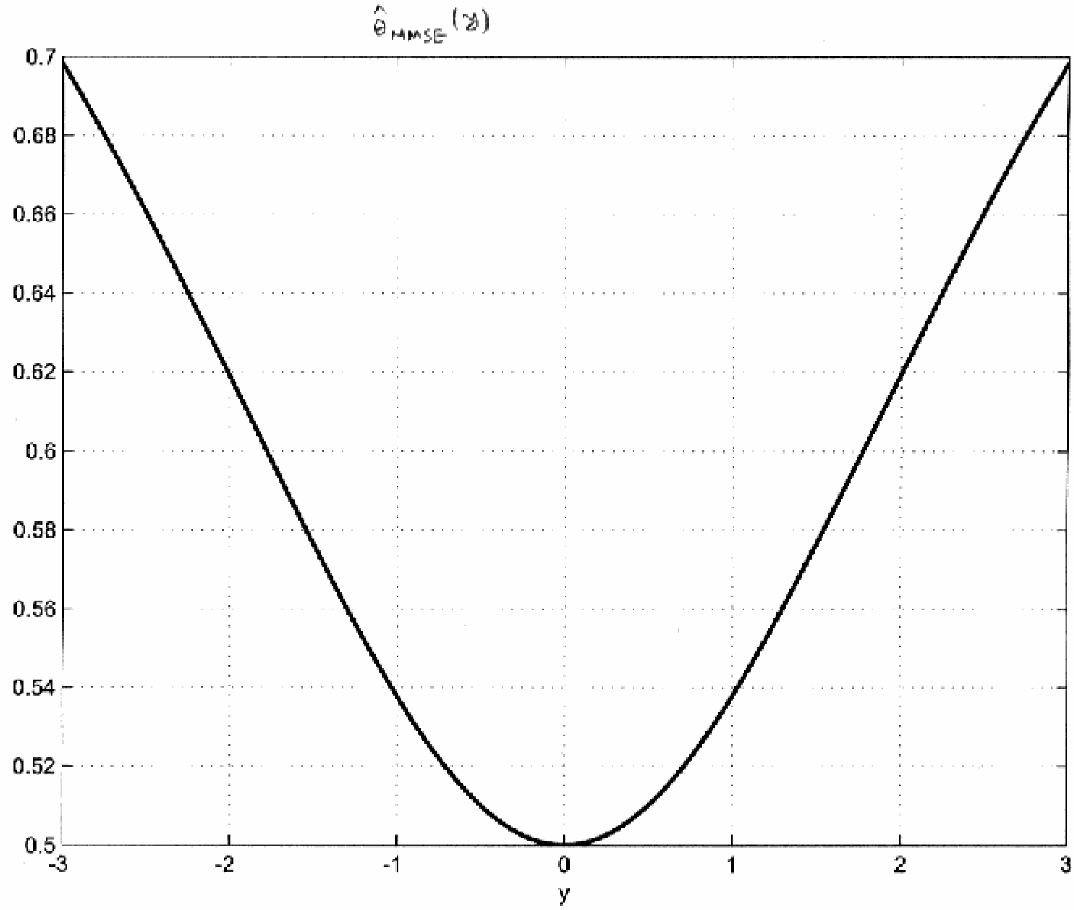

Fig. 2 Exercise 2

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3 $$w(\theta|y) = \frac{p_\theta(y)\,w(\theta)}{\int_{-\infty}^{\infty} p_\theta(y)\,w(\theta)\,d\theta}$$

$$\int_{-\infty}^{\infty} p_\theta(y)\,w(\theta)\,d\theta = \frac{\alpha}{y!}\int_0^{\infty} \theta^y e^{-(1+\alpha)\theta}\,d\theta = \frac{\alpha}{(1+\alpha)^{y+1}\,y!}\int_0^{\infty} \theta^y e^{-\theta}\,d\theta$$

Gamma function: $\Gamma(t+1) = \int_0^{\infty} \theta^t e^{-\theta}\,d\theta$

$$\int_{-\infty}^{\infty} p_\theta(y)\,w(\theta)\,d\theta = \frac{\alpha\,\Gamma(y+1)}{y!\,(1+\alpha)^{y+1}}$$

$$\theta \geq 0 \Rightarrow w(\theta|y) = \frac{\frac{\alpha}{y!}\,\theta^y e^{-(1+\alpha)\theta}}{\frac{\alpha\,\Gamma(y+1)}{y!\,(1+\alpha)^{y+1}}} = \frac{(1+\alpha)^{y+1}}{\Gamma(y+1)}\,\theta^y e^{-(1+\alpha)\theta}$$

$$\theta < 0 \Rightarrow w(\theta|y) = 0$$

MMSE Estimator:

$$\hat{\theta}_{MMSE}(y) = E\{\theta \mid Y=y\} = \int_{-\infty}^{\infty} \theta\,w(\theta|y)\,d\theta = \frac{(1+\alpha)^{y+1}}{\Gamma(y+1)}\int_0^{\infty} \theta^{y+1} e^{-(1+\alpha)\theta}\,d\theta$$

$$= \frac{\Gamma(y+2)}{(1+\alpha)\,\Gamma(y+1)} \quad \text{(see Fig. 3)}$$

MAP Estimator:

$$\frac{\partial}{\partial\theta}\ln w(\theta|y) = 0 \Rightarrow \frac{\partial}{\partial\theta}\left\{y\ln\theta - (1+\alpha)\theta\right\} = 0$$

$$\Rightarrow \frac{y}{\theta} - (1+\alpha) = 0 \Rightarrow \hat{\theta}_{MAP} = \frac{y}{1+\alpha}$$

MMAE Estimator: $\int_0^{\hat{\theta}_{ABS}(y)} w(\theta|y)\,d\theta = 0.5 \Rightarrow \hat{\theta}_{ABS}(y)$ can be numerically computed.

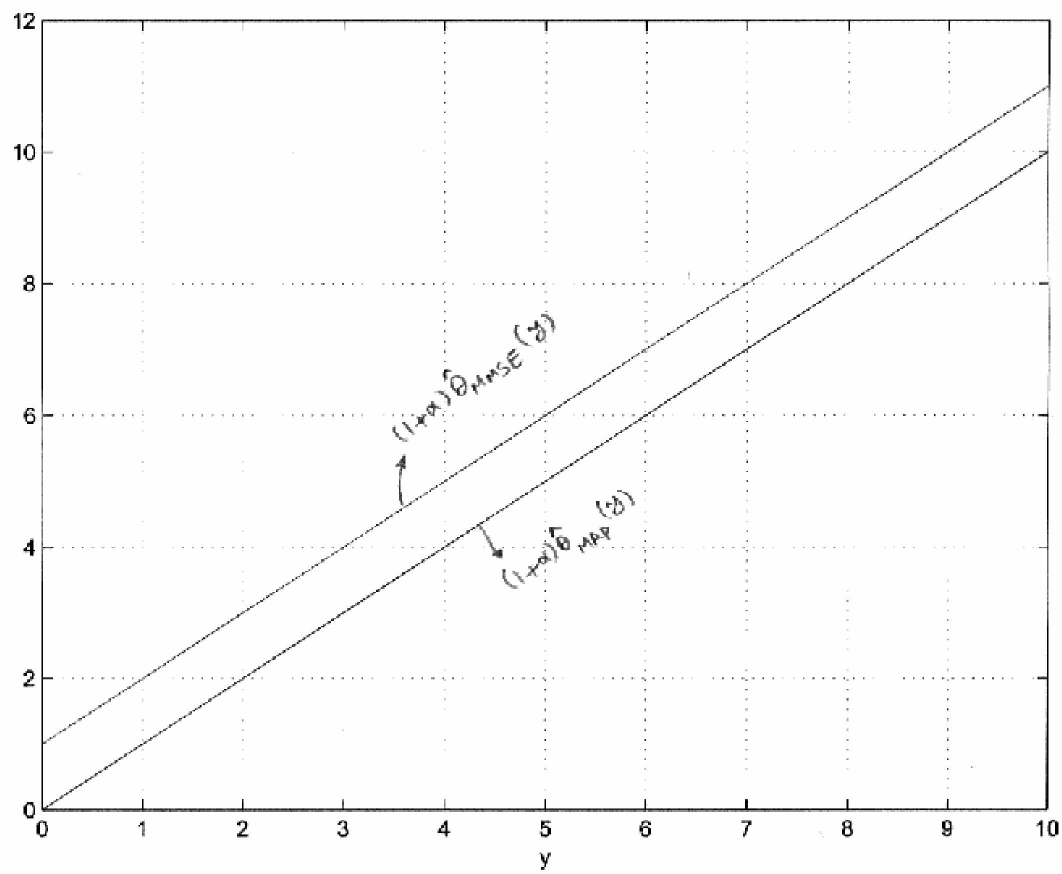

Fig. 3. Exercise 3

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4. $N \sim \mathcal{N}(0, \sigma^2)$

$$P_\theta(y) = p(y|\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\theta)^2/2\sigma^2}$$

$$p(y) = \sum_{\theta \in \{\pm 1\}} P_\theta(y)\, w(\theta) = \frac{1}{2} P_{+1}(y) + \frac{1}{2} P_{-1}(y)$$

$$= \frac{1}{2\sigma\sqrt{2\pi}} \left[e^{-(y-1)^2/2\sigma^2} + e^{-(y+1)^2/2\sigma^2} \right]$$

$$= \frac{1}{2\sigma\sqrt{2\pi}} e^{-(y^2+1)/2\sigma^2} \left(e^{y/\sigma^2} + e^{-y/\sigma^2} \right)$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-(y^2+1)/2\sigma^2} \cosh(y/\sigma^2)$$

$$w(\theta|y) = \frac{P_\theta(y)\, w(\theta)}{p(y)} = \frac{e^{y\theta/\sigma^2}}{2\cosh(y/\sigma^2)}$$

(a) <u>MMSE Estimator</u>:

$$\hat{\theta}_{MMSE}(y) = E\{\Theta | Y = y\} = \sum_{\theta \in \{\pm 1\}} \theta\, w(\theta|y) = \frac{1}{2\cosh(y/\sigma^2)} \left(e^{y/\sigma^2} - e^{-y/\sigma^2} \right)$$

$$= \frac{\sinh(y/\sigma^2)}{\cosh(y/\sigma^2)} = \tanh(y/\sigma^2) \qquad \text{(See Fig. 4)}$$

<u>MAP Estimator</u>:

$$\hat{\theta}_{MAP} = \arg\max_{\theta \in \{\pm 1\}} w(\theta|y) = \begin{cases} 1, & y > 0 \\ -1, & y < 0 \end{cases} \qquad \text{(see Fig. 4)}$$

(b) Fig. 4 shows when $\sigma^2 \ll 1$, these two estimates are approximately equal

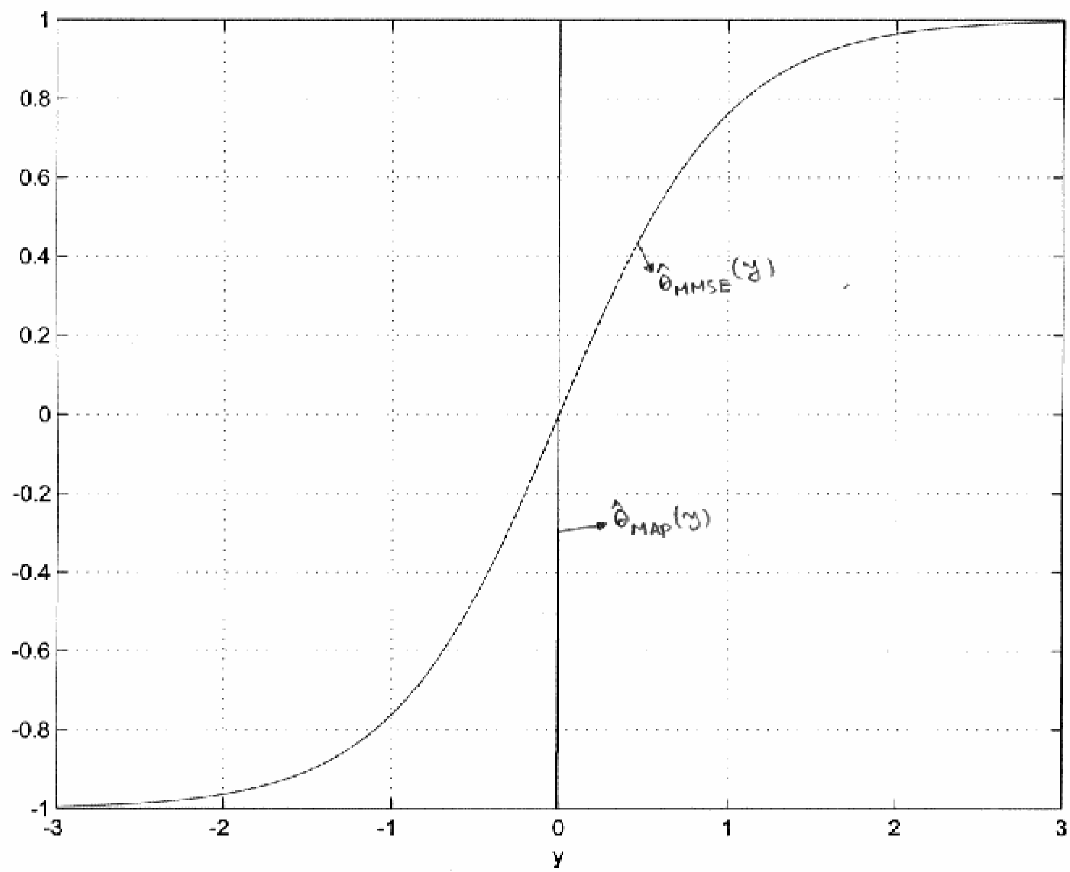

Fig. 4. Exercise 4

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5. $p(y) = \int_{-\infty}^{\infty} p_\theta(y)\, w(\theta)\, d\theta = \int_0^\infty \frac{1}{2} e^{-\theta - |y-\theta|} d\theta$

If $y<0 \Rightarrow p(y) = \frac{1}{2}\int_0^\infty e^{-\theta-(\theta-y)} d\theta = \frac{1}{2} e^{y} \int_0^\infty e^{-2\theta} d\theta$

$$= \frac{1}{2} e^{y} \left[-\frac{1}{2} e^{-2\theta} \right]_0^\infty = \frac{1}{4} e^{y}$$

If $y \geq 0 \Rightarrow p(y) = \frac{1}{2}\int_0^y e^{-\theta-(y-\theta)} d\theta + \frac{1}{2}\int_y^\infty e^{-\theta-(\theta-y)} d\theta$

$$= \frac{1}{2} y e^{-y} + \frac{1}{2} e^{y} \int_y^\infty e^{-2\theta} d\theta$$

$$= \frac{1}{2} y e^{-y} + \frac{1}{2} e^{y} \left[-\frac{1}{2} e^{-2\theta} \right]_y^\infty$$

$$= \frac{1}{2} y e^{-y} + \frac{1}{4} e^{-y} = \frac{1}{4}(1+2y) e^{-y}$$

$$\therefore \quad p(y) = \begin{cases} \frac{1}{4} e^{y}, & y<0 \\ \frac{1}{4}(1+2y) e^{-y}, & y \geq 0 \end{cases}$$

$$w(\theta|y) = \frac{p_\theta(y)\, w(\theta)}{p(y)}$$

If $\theta \geq 0,\ y<0 \Rightarrow w(\theta|y) = \dfrac{\frac{1}{2} e^{-|y-\theta|}\, e^{-\theta}}{\frac{1}{4} e^{y}} = 2 e^{-y-\theta-(\theta-y)} = 2 e^{-2\theta}$

If $\theta \geq 0,\ y \geq 0 \Rightarrow w(\theta|y) = \dfrac{\frac{1}{2} e^{-|y-\theta|}\, e^{-\theta}}{\frac{1}{4}(1+2y) e^{-y}} = \dfrac{2}{1+2y} e^{y-\theta-|y-\theta|}$

<u>MMSE Estimator</u> :

$$\hat{\theta}_{MMSE}(y) = E\{\Theta | Y=y\} = \int_{-\infty}^{\infty} \theta\, w(\theta|y)\, d\theta$$

If $y<0 \Rightarrow \hat{\theta}_{MMSE}(y) = 2\int_0^\infty \theta e^{-2\theta} d\theta = -\int_0^\infty \theta\, d e^{-2\theta} = -\left[\theta e^{-2\theta}\right]_0^\infty + \int_0^\infty e^{-2\theta} d\theta$

$$= \left[-\frac{1}{2}e^{-2\theta}\right]_0^\infty = 1/2$$

If $y \geqslant 0 \Rightarrow \hat{\theta}_{MMSE}(y) = \frac{2}{1+2y}\int_0^\infty e^{y-\theta-|y-\theta|}\,d\theta$

$$= \frac{2}{1+2y}\int_0^y e^{y-\theta-(y-\theta)}\,d\theta + \frac{2}{1+2y}\int_y^\infty e^{y-\theta-(\theta-y)}\,d\theta$$

$$= \frac{2y}{1+2y} + \frac{2e^{2y}}{1+2y}\int_y^\infty e^{-2\theta}\,d\theta$$

$$= \frac{2y}{1+2y} + \frac{2e^{2y}}{1+2y}\left[-\frac{1}{2}e^{-2\theta}\right]_y^\infty$$

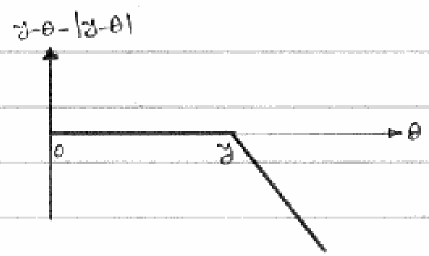
$$= \frac{2y+1}{1+2y} = 1$$

$$\therefore\ \hat{\theta}_{MMSE} = \begin{cases} 1/2, & y<0 \\ 1, & y \geqslant 0 \end{cases}$$

<u>MAP Estimator</u> :

$y<0 \Rightarrow \hat{\theta}_{MAP}(y) = \underset{\theta \geqslant 0}{\arg\max}\left\{2e^{-2\theta}\right\} = 0$

$y \geqslant 0 \Rightarrow \hat{\theta}_{MAP}(y) = \underset{\theta \geqslant 0}{\arg\max}\left\{\frac{2}{1+2y}e^{y-\theta-|y-\theta|}\right\} = \underset{\theta \geqslant 0}{\arg\max}\left\{y-\theta-|y-\theta|\right\}$



$0 \leq \theta \leq y \Rightarrow y-\theta-|y-\theta| = y-\theta-(y-\theta) = 0$

$\theta \geqslant y \Rightarrow y-\theta-|y-\theta| = y-\theta-(\theta-y) = 2y-2\theta$

$$\therefore\ \hat{\theta}_{MAP}(y) = \begin{cases} 0, & y<0 \\ \text{anything in } [0, y], & y \geqslant 0 \end{cases}$$

6. $N_1, N_2 \sim \mathcal{N}(0,0,1,1;\rho)$

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Given Θ, Y_1 and Y_2 are jointly normal with the following information:

mean $E\{Y_k \mid \Theta=\theta\} = \frac{1}{\sqrt{\theta}} E\{N_k\} = 0, \quad k=1,2$

variance $E\{Y_k^2 \mid \Theta=\theta\} = \frac{1}{\theta} E\{N_k^2\} = \frac{1}{\theta}, \quad k=1,2$

covariance $E\{Y_1 Y_2 \mid \Theta=\theta\} = \frac{1}{\theta} E\{N_1 N_2 \mid \Theta=\theta\}$

correlation coefficient $r_{Y_1,Y_2|\Theta=\theta} = \frac{\sigma_{Y_1,Y_2|\theta}}{\sigma_{Y_1|\theta}\,\sigma_{Y_2|\theta}} = \frac{1/\theta\, E\{N_1 N_2\}}{1/\theta} = r_{N_1,N_2} = \rho$

$\therefore\ Y_1, Y_2 \mid \Theta=\theta \sim \mathcal{N}\left(0,0,\frac{1}{\sqrt{\theta}},\frac{1}{\sqrt{\theta}};\rho\right)$

$$P_\theta(y_1,y_2) = \frac{\theta}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{\theta}{2(1-\rho^2)}\left(y_1^2+y_2^2-2\rho y_1 y_2\right)\right\}$$

$$P(y_1,y_2) = \int_{-\infty}^{\infty} P_\theta(y_1,y_2)\, w(\theta)\, d\theta$$

$$A \triangleq \frac{y_1^2+y_2^2-2\rho y_1 y_2}{2(1-\rho^2)} \Rightarrow P(y_1,y_2) = \frac{1}{2\pi\alpha\sqrt{1-\rho^2}} \int_0^\alpha \theta\, e^{-A\theta}\, d\theta$$

$$= -\frac{1}{2\pi\alpha A\sqrt{1-\rho^2}} \int_0^\alpha \theta\, d e^{-A\theta}$$

$$= -\frac{1}{2\pi\alpha A\sqrt{1-\rho^2}} \left[\theta e^{-A\theta}\right]_0^\alpha + \frac{1}{2\pi\alpha A\sqrt{1-\rho^2}} \int_0^\alpha e^{-A\theta}\, d\theta$$

$$= -\frac{e^{-A\alpha}}{2\pi A\sqrt{1-\rho^2}} + \frac{1}{2\pi\alpha A\sqrt{1-\rho^2}} \left[-\frac{1}{A} e^{-A\theta}\right]_0^\alpha$$

$$= -\frac{e^{-A\alpha}}{2\pi A\sqrt{1-\rho^2}} + \frac{1}{2\pi\alpha A^2\sqrt{1-\rho^2}} \left(1-e^{-A\alpha}\right)$$

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$$\therefore\ P(y_1,y_2)=\frac{1}{2\pi A\sqrt{1-\rho^2}}\left\{\frac{1}{\alpha A}\left(1-e^{-A\alpha}\right)-e^{-A\alpha}\right\}$$

$$\theta\in[0,\alpha]\Rightarrow w(\theta|y_1,y_2)=\frac{P_\theta(y_1,y_2)\,w(\theta)}{P(y_1,y_2)}$$

$$\theta\notin[0,\alpha]\Rightarrow w(\theta|y_1,y_2)=0$$

(a) MMSE Estimator:

$$\hat{\theta}_{MMSE}(y)=E\{\Theta|Y_1=y_1,Y_2=y_2\}=\int_{-\infty}^{\infty}w(\theta|y_1,y_2)\,\theta\,d\theta$$

$$=\frac{1}{2\pi\alpha\sqrt{1-\rho^2}}\times\frac{1}{P(y_1,y_2)}\int_0^{\alpha}\theta^2e^{-\theta A}\,d\theta$$

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7. $\theta \sim U(0, 1)$

$$P_\theta(y) = P_N(y-\theta) = \begin{cases} e^{-(y-\theta)}, & y \geq \theta \\ 0, & y \leq \theta \end{cases}$$

$$p(y) = \int_{-\infty}^{\infty} P_\theta(y)\, w(\theta)\, d\theta = \int_0^1 P_\theta(y)\, d\theta$$

1) $y < 0 \Rightarrow p(y) = 0$

2) $0 \leq y \leq 1 \Rightarrow p(y) = \int_0^y e^{-(y-\theta)} d\theta = e^{-y}\left[e^{\theta}\right]_0^y = 1 - e^{-y}$

3) $y > 1 \Rightarrow p(y) = \int_0^1 e^{-(y-\theta)} d\theta = e^{-y}\left[e^{\theta}\right]_0^1 = (e-1)\, e^{-y}$

$$\therefore\ p(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-y}, & 0 \leq y \leq 1 \\ (e-1)e^{-y}, & y > 1 \end{cases}$$

$$w(\theta|y) = \frac{P_\theta(y)\, w(\theta)}{p(y)}$$

$\theta \notin (0, 1) \Rightarrow w(\theta|y) = 0$

$\theta \in (0,1) \Rightarrow$ ① $0 \leq y \leq 1$: $w(\theta|y) = \dfrac{e^{-(y-\theta)}}{1-e^{-y}} = \dfrac{e^{-y}}{1-e^{-y}}\, e^{\theta}, \quad 0 \leq \theta \leq y$

② $y > 1$: $w(\theta|y) = \dfrac{e^{-(y-\theta)}}{(e-1)\, e^{-y}} = \dfrac{1}{e-1}\, e^{\theta}, \quad 0 \leq \theta \leq 1$

$$\therefore\ \boxed{\theta \in (0,1),\ y \geq 0 \Rightarrow w(\theta|y) = \frac{1}{e^{\min(1,y)} - 1}\, e^{\theta}}$$

<u>MMSE Estimator</u> :

$$\hat{\theta}_{MMSE}(y) = E\{\theta | Y = y\} = \int_{-\infty}^{\infty} \theta\, w(\theta|y)\, d\theta$$

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$$y \geq 0 \Rightarrow \hat{\theta}_{MMSE}(y) = \frac{1}{e^{\min(1,y)}-1} \int_0^1 \theta e^{\theta} d\theta$$

$$= \frac{1}{e^{\min(1,y)}-1} \int_0^1 \theta \, de^{\theta}$$

$$= \frac{1}{e^{\min(1,y)}-1} \left[\theta e^{\theta}\right]_0^1 - \frac{1}{e^{\min(1,y)}-1} \int_0^1 e^{\theta} d\theta$$

$$= \frac{e}{e^{\min(1,y)}-1} - \frac{e-1}{e^{\min(1,y)}-1} = \frac{1}{e^{\min(1,y)}-1}$$

MMAE Estimator:

$$\int_0^{\hat{\theta}_{ABS}(y)} w(\theta|y) d\theta = 0.5 \Rightarrow \frac{1}{e^{\min(1,y)}-1} \int_0^{\hat{\theta}_{ABS}(y)} e^{\theta} d\theta = 0.5$$

$$\Rightarrow e^{\hat{\theta}_{ABS}(y)} - 1 = 0.5\left[e^{\min(1,y)} - 1\right]$$

$$\Rightarrow \hat{\theta}_{ABS}(y) = \ln\{e^{\min(1,y)} + 1\} - \ln 2$$

MAP Estimator:

$$\hat{\theta}_{MAP}(y) = \arg\max_{0<\theta<1} \{w(\theta|y)\} = 1$$

(See Fig. 5)

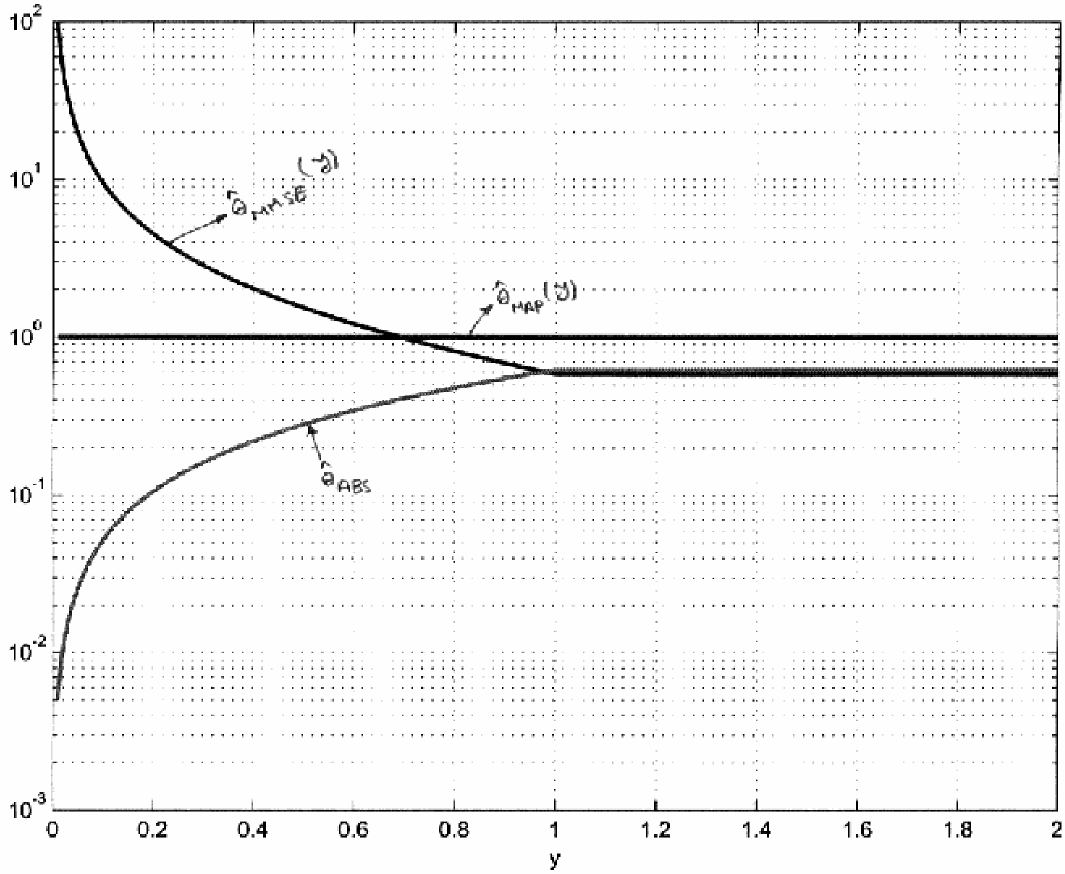

Fig. 5. Exercise 7