ECE 7251: Signal Detection and Estimation

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Lecture 23, 3/1/01:

Chernoff Bounds (Theory)

The Setup

- General purpose likelihood ratio test

$$\frac{p(y \mid H_1)}{p(y \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} l$$

- Consider the *log*likelihood ratio test

$$L \equiv \ln \Lambda = \ln \frac{p(y \mid H_1)}{p(y \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \ln l \equiv g$$

- Conditional error probabilities:

$$P_D = \int_g^\infty p_{L|H_1}(l \mid H_1)dl, \; P_{FA} = \int_g^\infty p_{L|H_0}(l \mid H_0)dl$$

The Problem

- Alas, it is often difficult, if not impossible, to find simple formulas for

$$p_{L|H_1}(l \mid H_1), \; p_{L|H_0}(l \mid H_0)$$

- Makes computing probabilities of detection and false alarm difficult
  - Could use Monte Carlo simulations, but those are cumbersome
  - Alternative: find easy to compute, analytic <u>bounds</u> on the error probabilities
- Discussion based on Van Trees, pp. 116-125

A Moment Generating Function

$$\Phi_{L|H_0}(s) \equiv E[e^{sL} \mid H_0] = \int_{-\infty}^{\infty} e^{sl} p(l \mid H_0)dl, \; k = 0,1$$

$$= \int_y \exp[sL(y)]p(y \mid H_0)dy$$

$$= \int_y \exp\left[s \ln \frac{p(y \mid H_1)}{p(y \mid H_0)}\right] p(y \mid H_0)dy$$

$$= \int_y \left[\frac{p(y \mid H_1)}{p(y \mid H_0)}\right]^s p(y \mid H_0)dy$$

$$= \int_y p(y \mid H_1)^s p(y \mid H_0)^{1-s} dy$$

Tilted Densities

- Define a new random variable X_s (for various values of s) with density

$$p_{X_s}(x) \equiv \frac{e^{sx} p_{L|H_0}(x \mid H_0)}{\int_{-\infty}^{\infty} e^{sl} p_{L|H_0}(l \mid H_0)dl}$$

The Happy Mu Function

$$\mu(s) \equiv \ln \Phi_{L|H_0}(s) = \ln \int_{-\infty}^{\infty} e^{sL} p(l \mid H_0)dl$$

$$\dot{\mu}(s) = \frac{\int_{-\infty}^{\infty} L e^{sL} p(l \mid H_0)dl}{\int_{-\infty}^{\infty} e^{sL} p(l \mid H_0)dl} = E[X_s]$$

$$\ddot{\mu}(s) = \text{var}[X_s]$$ **Exercise:** Show this!

More Properties of the Mu Function

$$\dot{m}(0)=\frac{\int_{-\infty}^{\infty} Le^{0L}p(l\,|\,H_0)dl}{\int_{-\infty}^{\infty} e^{0L}p(l\,|\,H_0)dl}=E[L\,|\,H_0]$$

$$\dot{m}(1)=\frac{\int_{-\infty}^{\infty} L\frac{p(l\,|\,H_1)}{p(l\,|\,H_0)}p(l\,|\,H_0)dl}{\int_{-\infty}^{\infty} \frac{p(l\,|\,H_1)}{p(l\,|\,H_0)}p(l\,|\,H_0)dl}=E[L\,|\,H_1]$$

A Weird Way of Writing P_{FA}

- With $m(s)=\ln\Phi_{L|H_0}(s)=\ln\int_{-\infty}^{\infty} e^{sL}p(l\,|\,H_0)dl$
- Then $\int_{g}^{\infty}\exp[m(s)-sx]p_{X_s}(x)dx$

$$=\int_{g}^{\infty}\exp[m(s)]e^{-sx}\frac{e^{sx}p_{L|H_0}(x\,|\,H_0)}{\int_{-\infty}^{\infty}e^{sl}p_{L|H_0}(l\,|\,H_0)dl}dx$$

$$=\int_{g}^{\infty}p_{L|H_0}(x\,|\,H_0)dx=P_{FA}$$

Creating the Bound

$$P_{FA}=\int_{g}^{\infty}\exp[m(s)-sx]p_{X_s}(x)dx$$

$$=e^{m(s)}\int_{g}^{\infty}e^{-sx}p_{X_s}(x)dx\le e^{m(s)}\int_{g}^{\infty}e^{-sg}p_{X_s}(x)dx$$

$$=\exp[m(s)-sg]\int_{g}^{\infty}p_{X_s}(x)dx$$

$$\le\exp[m(s)-sg]$$

Find the Tightest Bound

$$P_{FA}\le\exp[m(s)-sg]$$

- We want the $s\geq0$ which makes the RHS as small as possible

$$\frac{d}{ds}[m(s)-sg]=\dot{m}(s)-g$$

$$\dot{m}(s)=g$$

- Assuming everything worked (things exist, equation for maximizing *s* solvable, etc.):

$$P_{FA}\le\exp[m(s)-s\dot{m}(s)]$$

Similar Analysis Bounds P_M

$$P_M\le\exp[m(s)+(1-s)g]$$

- We want the $s\leq1$ which makes the RHS as small as possible

$$\frac{d}{ds}[m(s)+(1-s)g]=\dot{m}(s)-g$$

$$\dot{m}(s)=g$$

- Assuming everything worked (things exist, equation for maximizing *s* solvable, etc.):

$$P_M\le\exp[m(s)+(1-s)\dot{m}(s)]$$

Putting It All Together

$$P_{FA}\le\exp[m(s)-s\dot{m}(s)]$$

$$P_M\le\exp[m(s)+(1-s)\dot{m}(s)]\qquad 0\le s\le1$$

$$\text{where } g=\dot{m}(s)$$

$$\dot{m}(0)\le g\le\dot{m}(1)$$

$$E[L\,|\,H_0]\le g\le E[L\,|\,H_1]$$

Why is this useful? *L* can often be easily described by its moment generating function

Case of Equal Costs and Equal Priors

- Let s_m satisfy $\dot{m}(s_m) = \gamma = 0$

$$P_e = \frac{1}{2}P_{FA} + \frac{1}{2}P_M$$

$$\leq \frac{1}{2}\exp[m(s_m) - s\dot{m}(s_m)] + \frac{1}{2}\exp[m(s) + (1-s_m)\dot{m}(s_m)]$$

$$P_e \leq \exp[m(s_m)]$$

Another Look at the Derivation

$$P_{FA} = e^{m(s)} \int_{\dot{m}(s)}^{\infty} e^{-sx} p_{X_s}(x)dx$$

$$= \exp[m(s) - s\dot{m}(s)] \int_{\dot{m}(s)}^{\infty} \exp[+s(\dot{m}(s) - x)] p_{X_s}(x)dx$$

$$= \exp[m(s) - s\dot{m}(s)] \int_0^{\infty} \exp[-s\sqrt{\ddot{m}(s)}z] p_Z(z)dz$$

$$\text{where } Z = \frac{X_s - E[X_s]}{\sqrt{\mathrm{var}[X_s]}} = \frac{X_s - \dot{m}(s)}{\ddot{m}(s)}$$

A Revelation About the Constant

$$\exp[m(s) - s\dot{m}(s)] \underbrace{\int_0^{\infty} \exp[-s\sqrt{\ddot{m}(s)}z] p_Z(z)dz}$$

Original Chernoff inequality was formed by replacing this with 1. We can get a tighter constant in some asymptotic cases.

Asymptotic Gaussian Approximation

- In some cases, Z approaches a Gaussian random variable as the number of samples n grows large (ex: data points i.i.d. with finite means and variances

$$\int_0^{\infty} \exp[-s\sqrt{\ddot{m}(s)}z] \frac{1}{\sqrt{2\pi}} \exp\left[\frac{z^2}{2}\right] dz$$

$$= \exp\left[\frac{s^2 \ddot{m}(s)}{2}\right] Q(s\sqrt{\ddot{m}(s)})$$

Exercise: Show this!

Yet Another Approximation

$$P_{FA} = \exp[m(s) - s\dot{m}(s)] \int_0^{\infty} \exp[-s\sqrt{\ddot{m}(s)}z] p_Z(z)dz$$

$$\approx \exp[m(s) - s\dot{m}(s)] \exp\left[\frac{s^2 \ddot{m}(s)}{2}\right] Q(s\sqrt{\ddot{m}(s)})$$

- If $s\sqrt{\ddot{m}(s)} > 3$, we can approximate Q using an upper bound $Q(a) \leq \frac{1}{a\sqrt{2\pi}} \exp\left[-\frac{a^2}{2}\right]$

$$P_{FA} \approx \frac{1}{\sqrt{2\pi s^2 \ddot{m}(s)}} \exp[m(s) + s\dot{m}(s)]$$

Similar Analysis Works for P_M

$$P_M \approx e^{m(s)+(1-s)\dot{m}(s)} \exp\left[\frac{(s-1)^2 \ddot{m}(s)}{2}\right] Q((1-s)\sqrt{\ddot{m}(s)})$$

- If $(1-s)\sqrt{\ddot{m}(s)} > 3$, we can approximate Q using the upper bound

$$P_M \approx \frac{1}{\sqrt{2\pi(1-s)^2 \ddot{m}(s)}} \exp[m(s) + (1-s)\dot{m}(s)]$$

Asymptotic Analysis for P_e

- For the case of equal priors and equal costs, if the conditions for the approximation for Q to be valid on the previous to slides holds, we have

$$P_{FA} \approx \frac{1}{2s_m(1-s_m)\sqrt{2p\ddot{m}(s_m)}}\exp[m(s_m)]$$

Exercise: Show this! (It's not hard)