ECE 7251: Signal Detection and Estimation

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Lecture 22, 2/27/02:

Neyman-Pearson Tests and ROC Curves

The Setup

- Parametric data models $p(y;H_0),\ p(y;H_1)$
- No prior on hypotheses
- Goal as usually stated: design test which maximizes P_D (equivalently, minimizes P_M) under the constraint

$$P_{FA} = a' \leq a$$

- This lecture covers the case of simple hypotheses; composite hypotheses are much more complicated and will be covered later

Lagrange Multiplier Approach

$$f = P_M + l\,(P_{FA} - a')$$

$$= \int_{\mathcal{Y}_0} p(y \mid H_1)dy + l\left[\int_{\mathcal{Y}_1} p(y \mid H_0)dy - a'\right]$$

$$= \int_{\mathcal{Y}_0} p(y \mid H_1)dy + l\left[\left(1 - \int_{\mathcal{Y}_0} p(y \mid H_0)dy\right) - a'\right]$$

$$= \int_{\mathcal{Y}_0} p(y \mid H_1)dy - l\,p(y \mid H_0)dy + l\,(1 - a')$$

Lagrange Mult. Approach Con't

$$f = \int_{\mathcal{Y}_0} p(y \mid H_1)dy - l\,p(y \mid H_0)dy + l\,(1 - a')$$

- To minimize f, set

$$\mathcal{Y}_0 = \{y : p(y \mid H_1) - l\,p(y \mid H_0) < 0\}$$

$$= \{y : \frac{p(y \mid H_1)}{p(y \mid H_0)} < l\}$$

- Again we find the likelihood ratio

$$\Lambda(y) = \frac{p(y \mid H_1)}{p(y \mid H_0)}$$

The Constraint

$$\int_{\{y:\Lambda(y)>l\}} p(y \mid H_0)dy = P_{FA} = a' \leq a$$

$$\int_{\{y:\Lambda(y)<l\}} p(y \mid H_1)dy = P_M$$

- If we increase l, P_{FA} goes down and P_M goes up
- If we increase l, P_{FA} goes up and P_M goes down
- Hence, to minimize P_M, choose l so that P_{FA} is as big as possible under the constraint

Solving for the Threshold

- Mission: Find the λ which satisfies

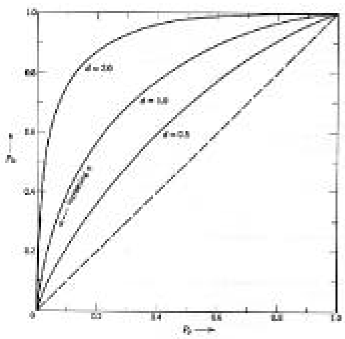
$$\int_{\{y:\Lambda(y)>l\}} p_{Y|H_0}(y \mid H_0)dy = P_{FA} = a$$

$$\int_{l}^{\infty} p_{\Lambda|H_0}(a \mid H_0)da = a$$

- Warning: we've been implicitly assuming P_{FA} is a continuous function of l (we'll return to this issue later)

Receiver Operating Characteristic

- Plot P_D vs. P_{FA} as we sweep the threshold



- Ex: ROC curve for Gaussian hypotheses, equal variances, different means

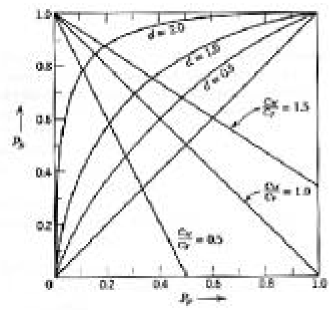
$$d = \frac{\sqrt{N}}{s}(m_1 - m_0)$$

(graph from Van Trees Vol. I, p. 38)

MinMax Lines on ROC Curves

- Recall MinMax tests equate conditional risks:

$$C_{FA} P_{FA} = C_M P_M = C_M (1 - P_D)$$



- MinMax point is intersection of ROC curve with the line

$$\frac{C_{FA}}{C_M} P_{FA} = 1 - P_D$$

(graph from Van Trees Vol. I, p. 45)

Example: Poisson Data

- Suppose we collect one data point and want to determine:

$$H_1 : y \sim Poisson(m_1)$$
$$H_0 : y \sim Poisson(m_0)$$

$$\frac{m_1^y \exp(-m_1)/y!}{m_0^y \exp(-m_0)/y!} \underset{H_0}{\overset{H_1}{\gtrless}} t$$

$$y[\ln m_1 - \ln m_0] - m_1 + m_0 \underset{H_0}{\overset{H_1}{\gtrless}} \ln t$$

$$y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\ln t + m_1 - m_0}{\ln m_1 - \ln m_0}$$

Poisson Example Con't

$$y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\ln t + m_1 - m_0}{\ln m_1 - \ln m_0} \equiv g$$

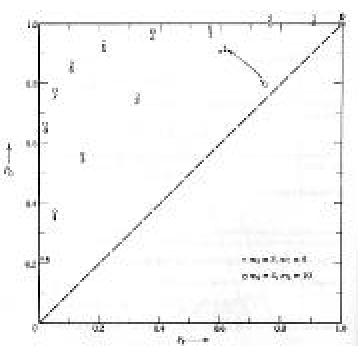
- Note y can only take integer values, so we may as well write

$$y \underset{H_0}{\overset{H_1}{\gtrless}} \equiv g_I, \text{ where } g_I \text{ is an integer}$$

- We have $P_D = 1 - \exp(-m_1) \sum_{y=0}^{g_I - 1} \frac{m_1^y}{y!}$

$$P_{FA} = 1 - \exp(-m_0) \sum_{y=0}^{g_I - 1} \frac{m_0^y}{y!}$$

ROC Curve for Poisson Test

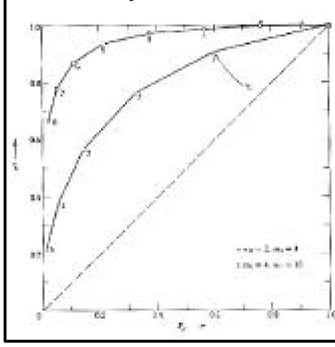


- Only certain points on the ROC graph are achievable using a normal LRT

(graph from Van Trees Vol. I, p. 42)

Randomization Fills in Gaps

- Randomly choose between two adjacent LRT tests



- Only needed since data took discrete values; probabilities of error were not continuous functions of the threshold

(graph from Van Trees Vol. I, p. 43)

Properties of ROC Curves

- ROC curves lie above the $P_D=P_{FA}$ line
  - A test is <u>unbiased</u> if its P_D is at least as big as its P_{FA}, i.e. $P_D \geq P_{FA}$
  - Note Neyman-Pearson LRT gives an unbiased test
- ROC curves are concave downward
- Slope of the ROC curve at a particular point equals the value of the threshold l required to achieve the P_D and P_{FA} at that point

ROC Property: Threshold=Slope

$$\frac{dP_D}{d\,l} = \frac{d}{d\,l}\int_l^{\infty} p_{\Lambda|H_1}(a\,|\,H_1)da = -p_{\Lambda|H_1}(\,l\,|\,H_1)\,(\;)$$

$$\frac{dP_{FA}}{d\,l} = \frac{d}{d\,l}\int_l^{\infty} p_{\Lambda|H_0}(a\,|\,H_0)da = -p_{\Lambda|H_0}(\,l\,|\,H_0)$$

$$\frac{dP_D/d\,l}{dP_{FA}/d\,l} = \underbrace{\frac{-p_{\Lambda|H_1}(\,l\,|\,H_1)}{-p_{\Lambda|H_0}(\,l\,|\,H_0)}}_{\text{We want to show this} = l} = \frac{P_D}{P_{FA}}$$

Showing That Thing We Want To Show

- Define

$$\Omega(\,l\,) \equiv \{y : \Lambda(y) \geq l\,\} = \{y : \frac{p_{Y|H_1}(\,l\,|\,H_1)}{p_{Y|H_0}(\,l\,|\,H_0)} \geq l\,\}$$

$$P_D(\,l\,) \equiv \Pr[\Lambda(y) \geq l\,|\,H_1\} = \int_{\Omega(\,l\,)} p_{Y|H_1}(y\,|\,H_1)dy$$

$$= \int_{\Omega(\,l\,)} \frac{p_{Y|H_1}(y\,|\,H_1)}{p_{Y|H_0}(y\,|\,H_0)} p_{Y|H_0}(y\,|\,H_0)dy$$

$$= \int_{\Omega(\,l\,)} \Lambda(y) p_{Y|H_0}(y\,|\,H_0)dy$$

More On Showing That Thing

$$P_D(\,l\,) = \int_{\Omega(\,l\,)} \Lambda(y) p_{Y|H_0}(y\,|\,H_0)dy$$

$$= \int_l^{\infty} a p_{\Lambda|H_0}(a\,|\,H_0)da$$

$$-p_{\Lambda|H_1}(\,l\,|\,H_1) \underset{(\;)}{=} \frac{dP_D(\,l\,)}{d\,l} = -l\, p_{\Lambda|H_0}(\,l\,|\,H_0)$$

$$l = \frac{p_{\Lambda|H_1}(\,l\,|\,H_1)}{p_{\Lambda|H_0}(\,l\,|\,H_0)}$$