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Exact lower bound for proportion of maximally embedded beamlet

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Abstract

We prove that the maximal inscribed beamlet is at least 1/7 of an arbitrary line segment (i.e., beam). The implication of this result is addressed.

Keywords: Multiscale geometric analysis; Beamlets; Detecting line segments

1. Introduction

Within a unit square $[0, 1)^2$, *beamlets* are a special collection of dyadically organized line segments, which present various scales, locations, orientations, and lengths. This collection is named the *beamlet dictionary*, and is described on various occasions, such as [1] — first appearance to the author's knowledge — and [2,3]. Beamlets are invented to facilitate geometric analysis of curvy features on a plane. More specifically, consider a curve which may be potentially embedded in a random field on a unit square, how do we evaluate the plausibility of the presence of such a curve?

We consider a simplified version of the above problem: The presence of a line segment (i.e., a beam)? For future convenience, we use *beam* instead of *line segment*.

The principle of Multiscale Geometric Detection (MGD) [4] works as follows. First of all, a test statistic is established. This statistic is based on some well-established statistical principles, e.g., the Generalized Likelihood Ratio Test (GLRT). Secondly, the statistics associated with the elements from a special set (in our case, this is the beamlet dictionary) are computed. There should be two advantages:

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(1) there is a efficient algorithm to compute, and (2) the cardinality of the special set has a significantly lower order than the set of all targets (e.g., all possible beams) that one would like to examine.

As an example, in the case of detecting the presence of a beam, in [4] it is proved that under a certain discretization, there are $O(n^4)$ beams, while there are just $O(n^2 \log(n))$ beamlets. Moreover, there is an $O(n^2 \log(n))$ algorithm to compute the values of GLRT-statistics for all beamlets [5], while due to the number of beams, such an order of complexity is impossible.

In a continuation of MGD, beamlets that are associated with *significant* test statistics are used as ‘seeds’; they are then extended to approximate any beam. It is proven from the asymptotic sense that such a scheme in theory can be both computationally fast and statistically optimal. Again we refer to the paper [4] for details.

Given the framework of MGD, it is of interest to study what is the minimal proportion of the maximally embedded beamlet, because it evaluates in the worst case how a “seed beamlet” can be correlated with a target beam. This motivates the study in this paper.

Note that to prove the asymptotic theory, all we need is the existence of such a constant. A quick-and-clever derivation of such a constant, which is not guaranteed to be the best, is provided in [4]. We may consider this paper as a refinement.

To describe our problem, we review the construction of a beamlet dictionary. There are three main steps.

- First of all, the unit square is dyadically partitioned: The unit square is equally partitioned into 2 by 2 dyadic sub-squares. Each sub-square is then equally divided into 2 by 2 smaller dyadic sub-squares. This process is repeated.
- Choose a small constant ε. Within a sub-square, starting from the northwest corner, we clockwise mark vertices along its boundary with inter-distance ε.
- In a sub-square, by connecting any pair of vertices, one gets a beamlet. The beamlet dictionary is the collection of all the beamlets that can be generated according to this procedure.

We skipped the stopping rule in the first step — dyadic partitioning. We always assume that the dyadic partitioning will proceed as needed.

On the other hand, in this paper, we assume that the constant ε is infinitely small, i.e., the beamlet can take any position on the boundary as its endpoints. The purpose of doing so is to simplify some future description.

Given a beam, its *maximally embedded beamlet* is the longest beamlet that is a fragment of this beam. As mentioned earlier, it is of interest to know what is the smallest possible proportion of the maximally embedded beamlet.

The main result of the paper is the following.

Theorem 1.1. *For any beam (i.e., line segment) B, and its decomposition into the beamlet dictionary, the longest beamlet b satisfies*

$$|b| \geq \frac{1}{7}|B|, \tag{1.1}$$

where $|b|$ and $|B|$ denote the lengths of beamlet b and beam B. The value $1/7$ cannot be increased, because there is a beam that is made by exactly seven equally long beamlets.

An illustration, which also is an extreme case of the above problem, is presented in Fig. 3.

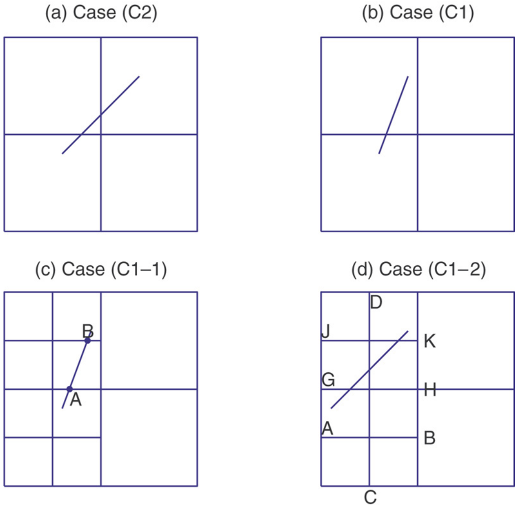

Fig. 1. Four possible positions of an underlying beam B.

The rest of the paper is organized as follows. Section 2 gives the detailed proof. Section 3 presents some discussion and concluding remarks.

2. Proof

At one step of dyadic partitioning, there is a horizontal partition line and a vertical partition line. Without loss of generality, we can assume that a beam B intersects with at least one of them. There are two possibilities:

(C1) B intersects with only one partition line, as in Fig. 1(b);
(C2) B intersects with both horizontal and vertical partition lines, as in Fig. 1(a).

We consider the case (C1) first. There are two dyadic squares that B occupies. We simultaneously partition the two squares. If no partitioning line intersects with B, this process is repeated. When B intersects with at least one of the partition line, there are following two possibilities.

(C1-1) Starting from the previously existing intersecting point, the next intersecting point is on one of the two parallel partition lines, as in Fig. 1(c).
(C1-2) Or the above is not true, and the underlying line segment B intersects with the partition line across the two squares. (For example in the situation illustrated in Fig. 1(b), this partition line is vertical.) An illustration is in Fig. 1(d).

In case (C1-1), there is a fragment of B which is at least one quarter of B. To see this, consider the segment AB in Fig. 1(c). Recall that between any two points on the boundary of a dyadic square, there is a beamlet. We proved (1.1).

In case (C1-2), if beam B does not intersect with any of the two parallel partition lines, then it can be classified into case (C2), which will be considered later. If B intersects with both parallel partition lines in the smaller sub-squares, then it is actually a case in (C1-1), which has been considered. If B intersects with only one of the two parallel partition lines, then it has to intersect with the partition line

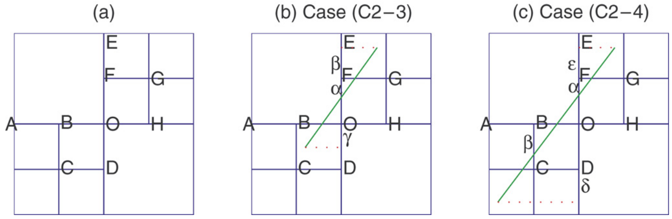

Fig. 2. Illustrations for case (C2).

on the same side as the intersecting point with the common partition line, which is the vertical line CD in Fig. 1(d); otherwise it belongs to case (C1-1). Apparently in case (C1-2), we only need to consider the configuration that is illustrated in Fig. 1(d). Since the total proportion of the two beamlets between lines GH and JK is at least $1/3$, the proportion of one of them has to be at least $1/6$. We proved (1.1).

Now we consider the case (C2). In case (C2), quadratic partitionings are simultaneously carried out in the two dyadic squares at the end of B. Eventually, B will intersect with one of the partition lines. Let Fig. 2(a) depict the result of the final partitioning. We have the following possibilities:

(C2-1) B intersects with either line segment AB or EF.
(C2-2) Case (C2-1) is not true and B intersects with either CD or GH.
(C2-3) Both case (C2-1) and (C2-2) are not true and B intersects with FG (resp. BC) but not BC (resp. FG).
(C2-4) Both case (C2-1) and (C2-2) are not true and B intersects with both BC and FG, but does *not* go beyond *at least one* of the two lines that are extended by CD and GH.
(C2-5) Similar to (C2-4), but B crosses the two lines that are extended by CD and GH.

Note that B must intersect with at least one of the next level partition lines. Hence the above categorization is comprehensive. We show that in all five cases, the theorem holds.

In case (C2-1), the segment of B in the square circumvented by $ABFE$ is at least $1/3$ of the length of B.

In case (C2-2), assume that B intersects with line segment CD. The segment of B between CD and OB must be at least $1/4$ of the length of B.

In case (C2-3), we assume that B does not intersect with BC. Since the total proportion of the two beamlets between lines OB and FG is at least $1/3$, the proportion of one of them has to be at least $1/6$.

In case (C2-4), we can use a similar argument as in case (C2-3).

In case (C2-5), the situation is a little more complicated. Let us first consider an extremal case that is depicted in Fig. 3. If the endpoints of B are infinitely close to points $(7/8, 1)$ and $(0, 1/8)$ but do not reach them, we can easily verify that the proportion of a maximum beamlet is nearly $1/7$. Following the notations in Fig. 3, we only need to prove that at least one of the intermediate segments (of B) — which are Z_1Z_2, Z_2Z_3, Z_3Z_4, Z_4Z_5, and Z_5Z_6 — is longer than $1/7$ of $|B|$. In Fig. 3, X and Y are two end points. Points Z_1, Z_2, Z_3, Z_4, Z_5, and Z_6 are on the beam B. Points A, B, C, D, E, F, G, H, J, and K are corners of dyadic sub-squares. It is easy to verify that

$$|AX| + |DZ_6| = |BZ_2| + |CZ_4|;$$

which, based on the similarity of the triangles that they reside, leads to

$$|XZ_1| + |YZ_6| = |Z_2Z_3| + |Z_4Z_5|.$$

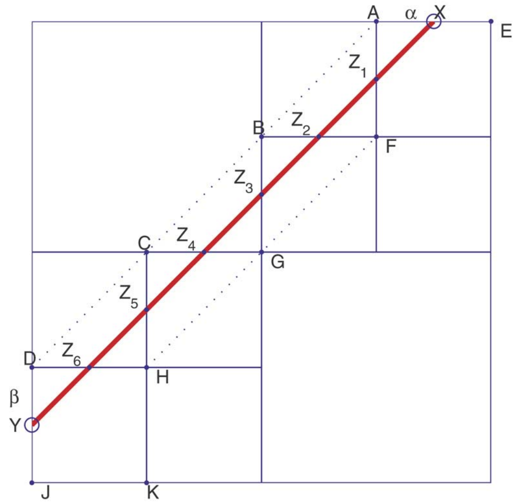

Fig. 3. Illustration for case (C2-5): Seven equally long beamlets superpose a beam.

Suppose all lengths $|Z_1Z_2|$, $|Z_2Z_3|$, $|Z_3Z_4|$, $|Z_4Z_5|$, and $|Z_5Z_6|$ are less than one seventh of $|B|$. We have

$$|XZ_1| + |YZ_6| < \frac{2}{7}|B|.$$

Hence we have

$$|XZ_1| + |Z_1Z_2| + |Z_2Z_3| + |Z_3Z_4| + |Z_4Z_5| + |Z_5Z_6| + |YZ_6| < |B|.$$

This is a contradiction. This contradiction shows that there is at least one intermediate segment of B whose proportion is no less than $1/7$. Note that it is possible that point Y is not in line segment DJ. For example, point Y could be on line segment JK. This situation can be handled by noticing the following facts:

(1) If Y is on JK, $|CZ_4|$ has to be at least $1/2$ of $|CG|$ — consider an extreme case of B traversing points A and J.
(2) $|Z_6H|$ can be no longer than $1/4$ of $|DH|$ — again consider an extreme case of B traversing the middle point of CG and point J.
(3) Considering similar triangles CZ_4Z_5 and HZ_6Z_5, we can conclude that

$$|Z_4Z_5| > 2|Z_5Z_6|.$$

Since $|Z_4Z_5| + |Z_5Z_6|$ is at least $1/4$ of $|B|$, we have

$$|Z_4Z_5| > \frac{2}{3} \cdot \frac{1}{4}|S| = \frac{1}{6}|B|.$$

Recall the extremal case that is given at the beginning of this paragraph; we proved the theorem in case (C2-5).

From all the above, Theorem 1.1 is proved.

3. Conclusion

We proved that the maximally embedded beamlet is at least $1/7$ of a beam. The proof is purely geometric, not requiring any prior knowledge. This result provides a useful constant in Multiscale Geometric Detection.

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