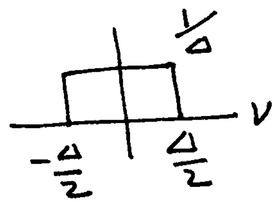
Problem 1 (16 pt. each)

A random variable X is uniformly distributed with mean 5 and variance 3.

a) Find the third moment of X;

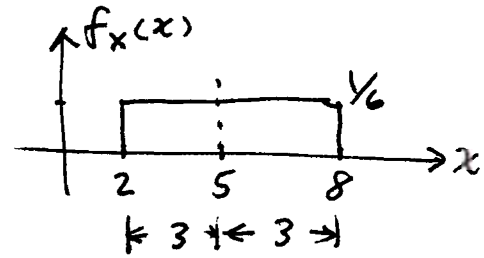
b) Let Y be a discrete random variable, statistically independent of X, with pdf $f_Y(y)=\frac{1}{2}\delta(y)+\frac{1}{2}\delta(y-6)$.
Find the pdf for Z, $Z=X+Y$.

$\sigma^2 = 3$



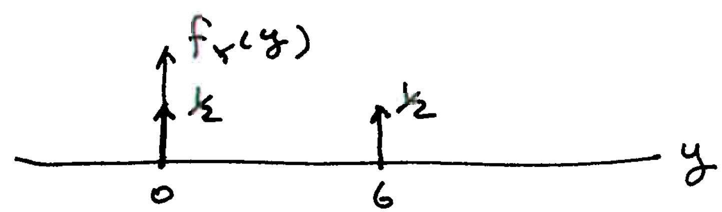
$\sigma_v^2 = \frac{\Delta^2}{12} = 3$, $\Delta^2 = 36$, $\Delta = 6$

Therefor $f_X(x) = \frac{1}{6}\left[u(x-2) - u(x-8)\right]$



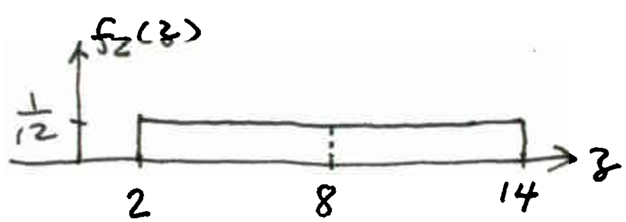
a) $\overline{X^3} = \int_2^8 x^3 \frac{1}{6}\, dx = \frac{x^4}{24}\Big|_2^8 = \frac{64^2-4^2}{24} = 170$

b) $f_Y(y) = \frac{1}{2}\delta(y) + \frac{1}{2}\delta(y-6)$



$Z = X + Y$

Since X and Y are independent, $f_Z(z) = f_X(z) \circledast f_Y(z)$



$f_Z(z) = \frac{1}{12}\left[u(z-2) - u(z-14)\right]$

Problem 2 (16 pt. each)

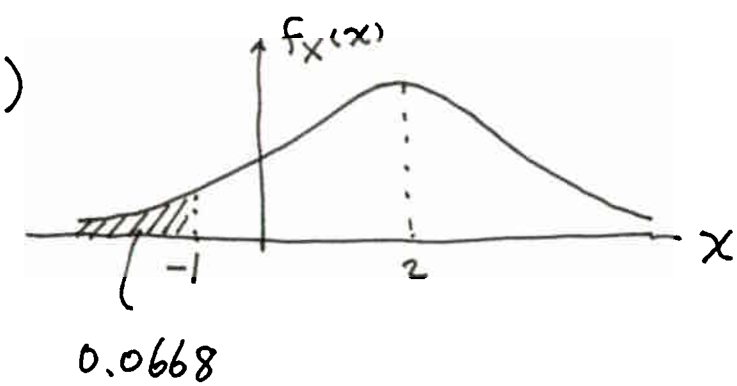
A Gaussian random current, X, has a probability of 0.5 of having value greater than 2. It also has a probability of 0.0668 of having a value less than or equal to -1.

a. Find the mean of X;
b. Find the variance of X.

a) X is Gaussian, $F_X(\bar{X}) = 0.5 = F_X(2)$

$\therefore \bar{X} = 2$

b)



$F_X(-1) = 0.0668$

$\Phi\left(\frac{-1-2}{\sigma_X}\right) = 1 - 0.0668$
$= 0.9332$

$\therefore \frac{-3}{\sigma_X} = -1.5$

$\sigma_X = 2$ and $\sigma_X^2 = 4$

Problem 3 (16 pt. each)

X and Y are independent random variables, each uniformly distributed in the interval (1,2). Let $Z = XY$ and $W = X^2$.

a) Find the joint probability density function $f_{Z,W}(z,w)$.

b) Find the correlation between Z and W.

a) $\begin{array}{l} Z = XY \\ W = X^2 \end{array} \Rightarrow \begin{array}{l} Y = \dfrac{Z}{\sqrt{W}} \\ X = \sqrt{W} \end{array}$, $\quad f_{X,Y}(x,y) = 1$ for $\begin{array}{l} 1 < x < 2 \\ 1 < y < 2 \end{array}$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial z} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial z} & \dfrac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & \dfrac{1}{2\sqrt{w}} \\ \dfrac{1}{\sqrt{w}} & -\dfrac{z w^{-\frac{3}{2}}}{2} \end{vmatrix} = -\frac{1}{2w}, \qquad |J| = \frac{1}{2w}, \quad w > 0$$

$$\therefore \quad f_{Z,W}(z,w) = \frac{1}{2w} \cdot 1 = \frac{1}{2w} \quad \text{for} \quad \begin{array}{l} 1 < \sqrt{w} < 2 \\ 1 < \dfrac{z}{\sqrt{w}} < 2 \end{array} \quad \text{or} \quad \begin{array}{l} 1 < w < 4 \\ \sqrt{w} < z < 2\sqrt{w} \end{array}$$

b) $R_{ZW} = E[ZW] = E[XY \cdot X^2] = E[X^3 Y] = E[X^3]E[Y]$

because X, Y are independant

$$E[X^3] = \int_1^2 x^3 dx = \left.\frac{x^4}{4}\right|_1^2 = \frac{16-1}{4} = \frac{15}{4}, \qquad E[Y] = \frac{2+1}{2} = \frac{3}{2}$$

$$\therefore \quad R_{ZW} = \frac{15}{4} \cdot \frac{3}{2} = \frac{45}{8}$$