Problem 1 (a=10 pts., b=20 pts.)

A chest drawer contains various resistors, as tabulated in terms of individual quantity, from three different plants, respectively. A resistor is to be randomly picked from the drawer.

a) List all the possible outcomes of the experiment;
b) Find x, the quantity of 10Ω resistor from Plant 2, such that the event of drawing a resistor from Plant 2 and the event of drawing a 10Ω resistor are statistically independent.

	Plant 1	Plant 2	Plant 3	Total
1 Ω	100	200	100	400
10 Ω	100	x	50	?
100 Ω	200	150	100	450
Total	400	?	250	?

a) (Plant 1, 1Ω), (Plant 2, 1Ω), (Plant 3, 1Ω)
(Plant 1, 10Ω), (Plant 2, 10Ω), (Plant 3, 10Ω)
(Plant 1, 100Ω), (Plant 2, 100Ω), (Plant 3, 100Ω)

b) For {Plant 2} and {10Ω} to be statistically independent,
Pr{(Plant 2, 10Ω)} must be equal to Pr{Plant 2} · Pr{10Ω}.

Total # of resistors = 1000 + x
Total # of resistors from Plant 2 = 350 + x
Total # of 10Ω resistors = 150 + x

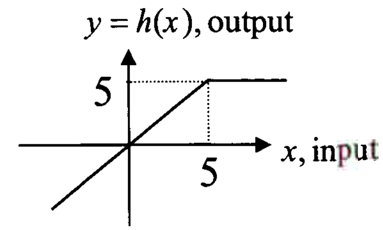
$$\frac{x}{1000+x} = \frac{350+x}{1000+x} \cdot \frac{150+x}{1000+x},$$

$$x^2 + 1000x = x^2 + 500x + 350 \cdot 150$$

$$500x = 350 \cdot 150 \quad \therefore \quad x = 105$$

Problem 2 (a=15 pts., b=20 pts.)

A hard clipper is a circuit that has the following input-output characteristic: $y = h(x) = \begin{cases} 5, & 5 < x \\ x, & x \le 5 \end{cases}$



The following figure shows an example of the effect of the hard clipper on a sine wave with amplitude greater than 5.

Now, suppose the input signal to the above clipper is a random variable, X, which is uniformly distributed in (-10, 10).

a) Find the probability density function of the output signal, Y.

b) Find the mean and the variance of Y.

a) $f_X(x) = \frac{1}{20}, \quad -10 < x < 10$

$\quad = 0, \quad$ elsewhere,

$P_r\{X > 5\} = \int_5^{10} \frac{1}{20} dx = \frac{1}{4} = P_r\{Y = 5\}$

For $-10 < y < 5$, $f_Y(y) = f_X(x) = \frac{1}{20}$

Therefore, $f_Y(y) = \frac{1}{20}[u(y+10) - u(y-5)] + \frac{1}{4}\delta(y-5)$

b) $E[Y] = \int_{-10}^{5} y \cdot \frac{1}{20} dy + \frac{1}{4} \cdot 5 = \frac{1}{20} \frac{y^2}{2}\Big|_{-10}^{5} + \frac{5}{4} = -0.625$

$E[Y^2] = \int_{-10}^{5} y^2 \cdot \frac{1}{20} dy + \frac{1}{4} \cdot (5)^2 = \frac{1}{20} \frac{y^3}{3}\Big|_{-10}^{5} + \frac{25}{4} = 25$

$\sigma_Y^2 = E[Y^2] - (E[Y])^2 = 25 - (0.625)^2 = 24.61 \quad < 33.3 = \sigma_X^2$

Mean of Y is -0.625

Variance of Y is 24.61

Problem 3 (a=20 pts., b=15 pts.)

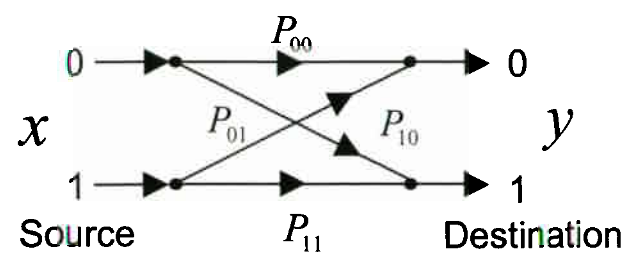
Consider a binary source and a transmission channel, as depicted. Let the source a priori probabilities be $\Pr(x=0)=p=0.8$ and $\Pr(x=1)=1-p=0.2$.

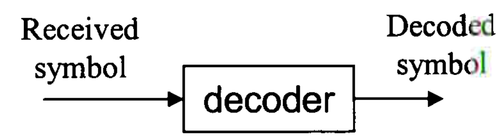
The channel, being non-ideal, is characterized by the four conditional probabilities: P_{00}, P_{01}, P_{10}, and P_{11} where $P_{ij}=\Pr(y=j\mid x=i),\quad i,j=0,1$.

Let $P_{00}=0.8, P_{01}=0.2, P_{10}=0.7$, and $P_{11}=0.3$, respectively.

a) Find the error probability at the destination if we consider (decode) the (unknown) source signal to be the same as the received signal;

b) Suggest a different decoding scheme that has a lower error probability than that in part a.





a) Error Probability $= p\,P_{01} + (1-p)\,P_{10} = 0.8\cdot 0.2 + 0.2\cdot 0.7 = 0.16 + 0.14 = 0.3$

b) If we set $Y=0$ as the decoded symbol regardless of the received symbol, then the error probability is

$(1-p)\,P_{10} + (1-p)\,P_{11} = 0.2$ which is less than 0.3 of part a.

In fact, given the source probability and the channel, the best single symbol decoder is the one that decodes every received symbol as 0.