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ECE 3075
HW #11
Solution

Problem 11-1
(6-3.4)

$$\mathcal{F}\{v(\tau)\} = \int_{-T}^{T}\left(1-\frac{|\tau|}{2}\right)e^{-j\omega\tau}d\tau = \int_{-T}^{T}e^{-j\omega\tau}d\tau - \frac{1}{2}\int_{-T}^{T}|\tau|e^{-j\omega\tau}d\tau$$

$$= \int_{-T}^{T}(\cos\omega\tau + \underbrace{j\sin\omega\tau}_{\text{odd fn.}})d\tau - \frac{1}{2}\int_{-T}^{T}(|\tau|\cos\omega\tau - \underbrace{j|\tau|\sin\omega\tau}_{\text{odd fn.}})d\tau$$

$$= \int_{-T}^{T}\cos\omega\tau\, d\tau - \frac{1}{2}\int_{-T}^{T}|\tau|\cos\omega\tau\, d\tau = 2\int_{0}^{T}\cos\omega\tau\, d\tau - \int_{0}^{T}\tau\cos\omega\tau\, d\tau$$

$$= \frac{2}{\omega}\sin\omega\tau\Big|_0^T - \left(\frac{\tau}{\omega}\sin\omega\tau - \frac{1}{\omega^2}\cos\omega\tau\right)\Big|_0^T$$

$$= \frac{(2-T)}{\omega}\sin\omega T - \frac{1}{\omega^2}\cos\omega T + \frac{1}{\omega^2}$$

For a valid $v(\tau)$, $\mathcal{F}\{v(\tau)\} \geq 0$: $0 \leq (2-T)\omega\sin\omega T - \cos\omega T + 1$

$$1 \geq \sqrt{1+(2-T)^2\omega^2}\cos(\omega T+\phi), \text{ where } \phi = \cos^{-1}\left(\frac{1}{\sqrt{1+(2-T)^2\omega^2}}\right)$$

$\Rightarrow \sqrt{1+(2-T)^2\omega^2} \leq 1$ for all ω.

$\Rightarrow \boxed{T=2}$ is the only solution.

Problem 11-2
(6-4.1)

(a) $\hat{X} = \frac{1}{21}\sum_{i=0}^{20} x_i = \boxed{0.0362}$

(b) $\hat{R}(0.01n) = \frac{1}{20-n+1}\sum_{k=0}^{20-n} X_k X_{k+n}$, $\boxed{\hat{R}(0) = 1.002}$ $\boxed{\hat{R}(0.01) = 0.581}$ $\boxed{\hat{R}(0.02) = 0.1669}$ $\boxed{\hat{R}(0.03) = -0.0460}$

(c) $\hat{R}(0.01n) = \frac{1}{20+1}\sum_{k=0}^{20-n} X_k X_{k+n}$, $\boxed{\hat{R}(0) = 1.002}$ $\boxed{\hat{R}(0.01) = 0.553}$ $\boxed{\hat{R}(0.02) = 0.151}$ $\boxed{\hat{R}(0.03) = -0.0387}$

Problem 11-3
(6-5.3)

(a) $(\bar{X})^2 = R_X(\infty) = 0$, $\overline{X^2} = R_X(0) = 10$, $\boxed{\sigma_X^2 = 10}$

(b) $\boxed{\sigma_X^2 = 10}$

(c) $\sigma_X^2 = 20 - 10 \boxed{= 10}$

$\boxed{\text{Problem 11-4}}$

(6-7.1)

$X(t)$ & $Y(t)$ independent $\Rightarrow R_{XY}(\tau) = R_{YX}(-\tau) = E[X(t)Y(t+\tau)] = E(X(t))E[Y(t+\tau)]$

$= E(X)E(Y) = \boxed{10}$

Since $R_X(\infty) = R_Y(\infty) = 0$.

let $W(t) = X(t) + Y(t)$

$Z(t) = X(t) - Y(t)$.

(a) $R_W(\tau) = R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{YX}(\tau) = R_X(\tau) + R_Y(\tau) = \boxed{25e^{-10|\tau|}\cos 100\pi\tau + 16\frac{\sin 50\pi\tau}{50\pi\tau}}$

(b) $R_Y(\tau) = R_X(\tau) + R_Y(\tau) - R_{XY}(\tau) - R_{YX}(\tau) = R_X(\tau) + R_Y(\tau) = \boxed{25e^{-10|\tau|}\cos 100\pi\tau + 16\frac{\sin 50\pi\tau}{50\pi\tau}}$

(same as (a))

(c) $R_{WY}(\tau) = E[(X(t)+Y(t))(X(t+\tau)-Y(t+\tau))] = R_X(\tau) - R_Y(\tau) - \cancel{R_{XY}(\tau)} + \cancel{R_{YX}(\tau)}$

$= \boxed{25e^{-10|\tau|}\cos 100\pi\tau - 16\frac{\sin 50\pi\tau}{50\pi\tau}}$

$R_{YW}(\tau) = R_{WY}(-\tau) = \boxed{R_{WY}(\tau)}$ ($R_{WY}(\cdot)$ even function)

(d) $E[(X(t)Y(t))(X(t+\tau)Y(t+\tau))] = E[X(t)X(t+\tau)]E[Y(t)Y(t+\tau)] = R_X(\tau)R_Y(\tau)$

$= \boxed{25e^{-10|\tau|}\cos 100\pi\tau \cdot 16\frac{\sin 50\pi\tau}{50\pi\tau}}$

$\boxed{\text{Problem 11-5}}$

(6-7.3)

(a) $R_{\dot{X}X}(\tau) = R_{X\dot{X}}(-\tau) = \frac{d}{d\tau}R_X(\tau)\Big|_{-\tau} = \frac{\tau\cos\tau - \sin\tau}{\tau^2}\Big|_{-\tau} = \boxed{\frac{-\tau\cos\tau + \sin\tau}{\tau^2}}$

(b) $R_{\dot{X}}(\tau) = -\frac{d^2R_X(\tau)}{d\tau^2} = \boxed{\frac{\tau^3\sin\tau + 2\tau^2\cos\tau - 2\tau\sin\tau}{\tau^4}}$