Problem 1 (10 pts)

A random variable X may have a characteristic function of the form $\Phi_X(u) = (0.4)^N e^{juN}[1 - pe^{ju}]^{-N}$. Determine the value of p, as a necessary condition, such that this is a legitimate characteristic function.

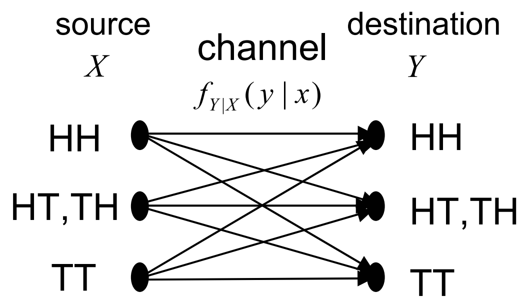
Answer: For the function to be a legitimate characteristic function, it is necessary that

$$\Phi_X(u)\Big|_{u=0} = (0.4)^N e^{juN}[1 - pe^{ju}]^{-N}\Big|_{u=0} = (0.4)^N (1-p)^{-N} = 1.$$ It follows then $1 - p = 0.4$, and thus $p = 0.6$.

Problem 2 (2 pts each answer, 30 pts total)

A source sends out signal according to the outcome of a 2-coin throwing experiment. The signal symbol set consists of HH, {HT or TH}, and TT – that is, no distinction between HT and TH is made. The two coins are biased; one has a head probability of 0.4 and the other 0.6. The signal is being sent through a channel, as depicted in the figure, which will cause a confusion between H and T with probability Pr{H|T}=0.1 and Pr {T|H}=0.05, respectively.

1. Find the probabilities of all symbols – TT, HH, {HT,TH} – at the destination.
2. Fill the blanks in the table.



Answer: $\Pr\{X = \text{HH}\} = 0.4 \times 0.6 = 0.24,$

$\Pr\{X = \text{HT or TH}\} = 0..4 \times 0.4 + 0.6 \times 0.6 = 0.52, \quad \Pr\{X = \text{TT}\} = 0.24$

$\Pr\{\text{H}\,|\,\text{T}\} = 0.1, \quad \Pr\{\text{T}\,|\,\text{H}\} = 0.05$

x	y	$f_{Y\|X}(y\|x)$	$f_{Y,X}(y,x)$	$f_{X\|Y}(x\|y)$
HH	HH	0.9025	0.2166	0.807
HH	HT, TH	0.095	0.0228	0.0444
HH	TT	0.0025	0.0006	0.0027
HT,TH	HT,TH	0.86	0.4472	0.8714
HT,TH	HH	0.095	0.0494	0.1841
HT,TH	TT	0.045	0.0234	0.1071
TT	TT	0.81	0.1944	0.8901
TT	TH,HT	0.18	0.0432	0.0842
TT	HH	0.01	0.0024	0.0089

$$\Pr\{Y = \text{HH}\} = \sum_x f_{Y|X}(y\,|\,x)\Pr\{x\} = 0.2684$$

$\Pr\{Y = \text{HT or TH}\} = 0.5132$

$\Pr\{Y = \text{TT}\} = 0.2184$

The entropy at the destination is

$H = 1.4826$ bits/symbol

Problem 3 (10 pts each, 40 pts total)

X and Y are independent random variables, uniformly distributed in the interval (0,1). Let $Z = XY$ and $W = X^2$

1. Find $f_{Z,W}(z,w)$.
2. Find the marginal density functions $f_Z(z)$ and $f_W(w)$.
3. Find the first three moments of Z.
4. Find the correlation between Z and W. Are they uncorrelated?

Answer:

Since X and Y are positive, only one solution in Z and W needs to be considered.

1. $Z = XY, \quad W = X^2$

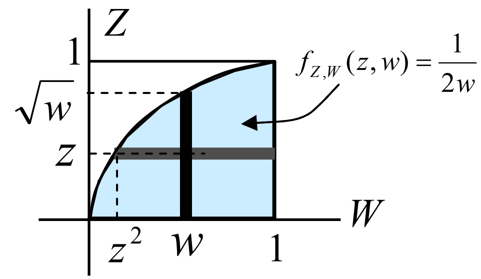
$X = W^{1/2}, \quad Y = Z/X = ZW^{-1/2}$

$$J = \begin{vmatrix} 0 & \frac{1}{2}W^{-1/2} \\ W^{-1/2} & \frac{-Z}{2}W^{-3/2} \end{vmatrix} = -\frac{1}{2W}$$

$$f_{Z,W}(z,w) = \frac{1}{2w} f_{X,Y}\left(\sqrt{w}, \frac{z}{\sqrt{w}}\right) = \frac{1}{2w} \qquad 0 < w < 1 \text{ and } 0 < z < \sqrt{w} \qquad \because Y = ZW^{-1/2} < 1$$

2. $$f_Z(z) = \int_{z^2}^{1} \frac{1}{2w} dw = \frac{1}{2} \ln w \Big|_{w=z^2}^{1} = -\ln(z), \qquad 0 < z < 1$$

$$f_W(w) = \int_0^{\sqrt{w}} \frac{1}{2w} dz = \frac{z}{2w} \Big|_{z=0}^{\sqrt{w}} = \frac{1}{2\sqrt{w}}, \quad 0 < w < 1$$



3. $E[X] = E[Y] = \frac{1}{2}, E[X^2] = E[Y^2] = \frac{1}{3}, E[X^3] = E[Y^3] = \frac{1}{4}$

$E[Z] = E[XY] = E[X]E[Y] = \frac{1}{4}$ $\qquad E[Z^2] = E[X^2]E[Y^2] = \frac{1}{9}$ $\qquad E[Z^3] = E[X^3]E[Y^3] = \frac{1}{16}$

4. $E[ZW] = E[X^3]E[Y] = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8} \neq E[Z]E[W]$

Therefore, they are not uncorrelated.

Problem 4 (10 pts each, 20 total)

Let U and V be two independent random variables with zero mean and variances σ_U^2 and σ_V^2, respectively.
Let X and Y be another two random variables with zero mean and variances σ_X^2 and σ_Y^2, respectively, and a correlation coefficient ρ. We shall call *X* and *Y* the signals while U and V the noise which are independent of the signals. The second moment of a random variable is customarily used to represent the power of the signal or noise that it represents. A signal, say *X*, is received at the destination as *X'=X+U*. Similarly, *Y'=Y+V*. The SNRs in the individually received signal *X'* and *Y'* are therefore σ_X^2/σ_U^2 and σ_Y^2/σ_V^2, respectively.

a. Show that the sum $W=U+V$ has a variance $\sigma_W^2=\sigma_U^2+\sigma_V^2$; that is, for independent random variables, the variance of sum is sum of variances.

b. If *X'* and *Y'* are added together with *X+Y* being the signal and *U+V* the noise, determine the SNR in terms of the individual variances and the correlation coefficient. Furthermore, if *X = Y* and $\sigma_U^2=\sigma_V^2$, compare the SNR in the individual signal *X'* and in the sum *X'+Y'*.

Answer:

a. $E[W]=E[U+V]=E[U]+E[V]=\overline{U}+\overline{V}$ $\qquad$ $E[W^2]=E[(U+V)^2]=\overline{U^2}+2\overline{U}\,\overline{V}+\overline{V^2}$

$$\sigma_W^2=\overline{W^2}-(\overline{W})^2=\overline{U^2}+2\overline{U}\,\overline{V}+\overline{V^2}-(\overline{U})^2-2\overline{U}\,\overline{V}-(\overline{V})^2=\left[\overline{U^2}-(\overline{U})^2\right]+\left[\overline{V^2}-(\overline{V})^2\right]=\sigma_U^2+\sigma_V^2$$

Note that the result is valid regardless of the means.

b. When the mean is zero, the variance (i.e., the 2nd central moment) equals the second moment.

$$E[(X+Y)^2]=\overline{X^2}+\overline{Y^2}+2E[XY]=\overline{X^2}+\overline{Y^2}+2\rho\sigma_X\sigma_Y=\sigma_X^2+\sigma_Y^2+2\rho\sigma_X\sigma_Y$$

$$E[(U+V)^2]=\overline{U^2}+\overline{V^2}+2E[UV]=\sigma_U^2+\sigma_V^2$$

Therefore, $SNR_{X'+Y'}=\dfrac{\sigma_X^2+\sigma_Y^2+2\rho\sigma_X\sigma_Y}{\sigma_U^2+\sigma_V^2}$

If $X=Y$, $\rho=\dfrac{E\{(X-\overline{X})^2\}}{\sigma_X^2}=1.$

With $\sigma_U^2=\sigma_V^2$, it follows that $SNR_{X'+Y'}=\dfrac{4\sigma_X^2}{2\sigma_U^2}=2SNR_{X'}$

In logarithm (dB) scale, $10\log_{10}SNR_{X'+Y'}=10\log_{10}2+10\log_{10}SNR_{X'}=3.01+10\log_{10}SNR_{X'}$, a gain of 3 dB.