ECE-3075A Random Signal
Fall 2003 - Final Exam

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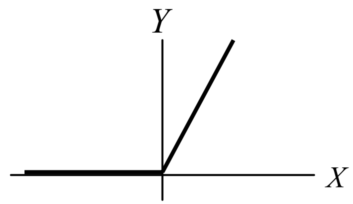
Problem 1 (12 pts, 6 ea.)

A Gaussian random variable X has a mean of 2 and a variance of 5.

A second random variable Y is related to X by $Y = X + |X|$.

a) Find the probability density function of Y;

b) A third random variable Z is a function of X and Y, $Z = \dfrac{Y}{X}$; find the first, the second and the third moment of Z.



$Y = X + |X|$.

If $X \le 0$, $Y = 0$, and $\Pr\{Y = 0\} = \Pr\{X \le 0\} = \Phi(-2/\sqrt{5}) = 1 - \Phi(0.89) = 1 - 0.8133 = 0.1867$

If $X > 0$, $Y = 2X$, then

$$X = Y/2, \quad |dx/dy| = 1/2, \quad f_Y(y) = \left|\frac{dx}{dy}\right| f_X\left(\frac{y}{2}\right) = \frac{1}{2}\frac{1}{\sqrt{2\pi \times 5}}\exp\left\{-\frac{[(y/2)-2]^2}{2\times 5}\right\}$$

$$= \frac{1}{2\sqrt{10\pi}}\exp\left\{-\frac{(y-4)^2}{40}\right\}$$

a)

$$f_Y(y) = \begin{cases} 0, & y < 0 \\ 0.1867\delta(y), & y = 0 \\ \dfrac{1}{2\sqrt{10\pi}}\exp\left\{-\dfrac{(y-4)^2}{40}\right\}, & y > 0 \end{cases} \quad \text{or} \quad f_Y(y) = 0.1867\delta(y) + \frac{1}{2\sqrt{10\pi}}\exp\left\{-\frac{(y-4)^2}{40}\right\}u(y)$$

b)

$$Z = \frac{Y}{X}, \quad X \ne 0$$

If $X > 0$, $Z = 2$, $\Pr\{Z = 2\} = \Pr\{X > 0\} = \Phi(0.89) = 0.8133$

If $X < 0$, $Z = 0$, $\Pr\{Z = 0\} = \Pr\{X < 0\} = 1 - \Phi(0.89) = 0.1867$

Therefore, $\overline{Z} = 0 \times 0.1867 + 2 \times 0.8133 = 1.6266$

$$\overline{Z^2} = 0 \times 0.1867 + 2^2 \times 0.8133 = 3.2532$$

$$\overline{Z^3} = 0 \times 0.1867 + 2^3 \times 0.8133 = 6.5064$$

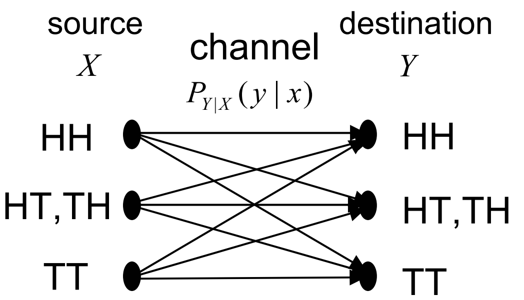
Problem 2 (24 pts, 6 ea.)

An information source sends out a signal which is a sequence of random symbols according to the outcome of a 2-coin tossing experiment. Each trial of the experiment thus produces a 2-tuple symbol. The signal set consists of HH, {HT or TH}, and TT – that is, no distinction between HT and TH is made. The two coins are independent but biased; one has a head probability of 0.4 and the other 0.7. The source signal is being sent through a channel, as depicted in the figure, which causes a confusion on the individual **elementary** symbol between H and T with probability

$$P\{\text{H(received)}|\text{T(sent)}\}=0.15=P(\text{H}|\text{T})$$

$$P\{\text{T(received)}|\text{H(sent)}\}=0.05=P(\text{T}|\text{H})$$

respectively. From the above elementary channel model, one can easily derive $P_{Y|X}(y|x)$.
For example, $P_{Y|X}(Y=\text{HT}\,|\,X=\text{HH})=P(\text{H}\,|\,\text{H})P(\text{T}\,|\,\text{H})$ since the two coins are independent.



a) Find the probabilities of all symbols – TT, HH, {HT,TH} – at the destination.
b) Given that either HT or TH has been received, find the probability that either HT or TH was sent.
c) Find the average information in bits in each 2-tuple symbol at the source.
d) Find the average received information in bits per 2-tuple symbol at the destination.

a) $\Pr(X=HH)=0.4\times 0.7=0.28,\quad \Pr(X=TT)=0.18,\quad \Pr(X\in\{HT,TH\})=0.4\times 0.3+0.6\times 0.7=0.54$

$$\Pr(Y=HH)=\Pr(Y=HH\,|\,X=HH)\Pr(X=HH)+\Pr(Y=HH\,|\,X=HT)\Pr(X=HT)$$
$$+\Pr(Y=HH\,|\,X=TH)\Pr(X=TH)+\Pr(Y=HH\,|\,X=TT)\Pr(X=TT)=0.3337$$

Similarly, $\Pr(Y=TT)=0.1537$, and $\Pr(Y\in\{HT,TH\})=0.5126$

b)
$$\Pr(X\in\{HT,TH\}\,|\,Y\in\{HT,TH\})=\frac{\Pr(X\in\{HT,TH\},Y\in\{HT,TH\})}{\Pr(Y\in\{HT,TH\})}$$

$$\Pr(X\in\{HT,TH\},Y\in\{HT,TH\})$$
$$=0.95\times 0.85\times 0.3\times 0.4+0.95\times 0.85\times 0.6\times 0.7+0.05\times 0.15\times 0.3\times 0.4+0.05\times 0.15\times 0.6\times 0.7=0.4401$$

$$\Pr(X\in\{HT,TH\}\,|\,Y\in\{HT,TH\})=\frac{\Pr(X\in\{HT,TH\},Y\in\{HT,TH\})}{\Pr(Y\in\{HT,TH\})}=\frac{0.4401}{0.5126}=0.8586$$

c) $$I=-\sum_i p_i\log_2 p_i=1.4395\quad (bits/symbol)$$

d) $$I=-\sum_i p_i\log_2 p_i=1.4379\quad (bits/symbol)$$

Problem 3 (18 pts, 6 ea.)

A stationary random process $X(t)$ has a power spectral density of

$S_x(\omega) = 32\pi\delta(\omega) + 16\pi[\delta(\omega-10) + \delta(\omega+10)] + 8\pi[\delta(\omega-20) + \delta(\omega+20)]$

a) Find the mean value of the process;

b) Find the variance of the process;

c) Find the autocorrelation function of the process.

a)
$$S_x(\omega) = 32\pi\delta(\omega) + 16\pi[\delta(\omega-10) + \delta(\omega+10)] + 8\pi[\delta(\omega-20) + \delta(\omega+20)]$$
$$= 2\pi(\overline{X})^2\delta(\omega) + 16\pi[\delta(\omega-10) + \delta(\omega+10)] + 8\pi[\delta(\omega-20) + \delta(\omega+20)]$$
$$(\overline{X})^2 = 32\pi / 2\pi = 16, \qquad \overline{X} = \pm 4$$

b)
$$\overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)d\omega = \int_{-\infty}^{\infty}\{16\delta(\omega) + 8[\delta(\omega-10) + \delta(\omega+10)] + 4[\delta(\omega-20) + \delta(\omega+20)]\}d\omega$$
$$= 16 + 8 + 8 + 4 + 4 = 40$$
$$\sigma_X^2 = \overline{X^2} - (\overline{X})^2 = 40 - 4^2 = 24$$

c)
$$R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)e^{j\omega\tau}d\omega$$
$$= 16 + 8e^{j10\tau} + 8e^{-j10\tau} + 4e^{j20\tau} + 4e^{j20\tau}$$
$$= 16 + 16\cos(10\tau) + 8\cos(20\tau)$$

Problem 4 (24 pts, 6 ea.)

A stationary random process has the form $X(t) = V(t) + 5\cos(10t + \Theta_1) + 10\sin(5t + \Theta_2)$ where $V(t)$ is an independent random process having an autocorrelation function of $R_V(\tau) = 10e^{-100|\tau|}$, Θ_1 and Θ_2 are random variables uniformly distributed in (0, 2π). All the random variables are statistically independent.

a) Find the mean value of the process;
b) Find the variance of the process;
c) Find the autocorrelation function of the process;
d) Find the spectral density of the process.

a) $$E\{X(t)\} = E\{V(t)\} + E\{5\cos(10t + \Theta_1)\} + E\{10\sin(5t + \Theta_2)\} = 0 + 0 + 0 = 0$$

b) $$\overline{X^2} = \overline{V^2} + \int_0^{2\pi} [5\cos(10t + \theta_1)]^2 \frac{1}{2\pi} d\theta_1 + \int_0^{2\pi} [10\sin(5t + \theta_2)]^2 \frac{1}{2\pi} d\theta_2$$

$$\overline{V^2} = R_V(0) = 10, \quad \int_0^{2\pi} \cos^2(10t + \theta_1) \frac{1}{2\pi} d\theta_1 = \frac{1}{2}, \quad \int_0^{2\pi} \sin^2(5t + \theta_2) \frac{1}{2\pi} d\theta_2 = \frac{1}{2}$$

$$\therefore \quad \overline{X^2} = 10 + \frac{25}{2} + \frac{100}{2} = 72.7 \quad \text{and thus,} \quad \sigma_X^2 = \overline{X^2} - (\overline{X})^2 = 72.5 - 0 = 72.5$$

c) $$R_X(\tau) = R_V(\tau) + \frac{25}{2}\cos(10\tau) + \frac{100}{2}\cos(5\tau) = 10e^{-100|\tau|} + 12.5\cos(10\tau) + 50\cos(5\tau)$$

due to the fact that all random variables are independent.

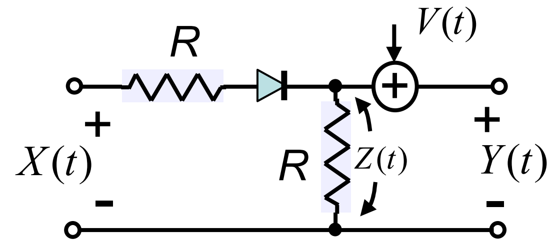
d) $$S_X(\omega) = \mathbf{F}\{R_X(\tau)\} = \mathbf{F}\{10e^{-100|\tau|}\} + \mathbf{F}\{12.5\cos(10\tau)\} + \mathbf{F}\{50\cos(5\tau)\}$$

$$= \frac{2000}{10^4 + \omega^2} + 12.5\pi[\delta(\omega + 10) + \delta(\omega - 10)] + 50\pi[\delta(\omega + 5) + \delta(\omega - 5)]$$

Problem 5 (24 pts, 8 ea.)

A random binary process $X(t)$ of full duty cycle (see Fig. 6-1, Section 6-2) has an amplitude ±8 and $t_a = 0.01$. The binary values are equally probable. It is applied to a half-wave rectifier circuit, as shown, which has an intrinsic additive noise $V(t)$ which is i.i.d. Gaussian with zero mean and a variance of 2.

a) Find the autocorrelation of $Y(t)$, i.e., $R_Y(\tau)$;
b) Find the spectral density of $Y(t)$, i.e., $S_Y(\omega)$;
c) Find the cross-correlation $R_{XY}(\tau)$.

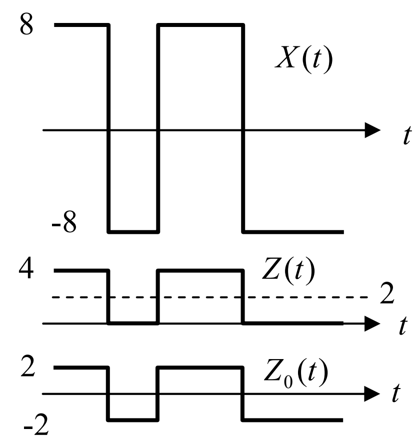


$Y(t) = Z(t) + V(t)$ $\qquad$ $Z(t)$ and $V(t)$ are independent.

As shown, – when $X(t) > 0$, $\quad Z(t) = X(t) \bullet \dfrac{R}{2R} = \dfrac{X(t)}{2} = 4$

when $X(t) < 0$, $\quad Z(t) = 0$

Furthermore, $Z(t) = A + Z_0(t)$ where A is the DC component, $A = 2$, and $Z_0(t)$ is a zero mean binary random process with amplitude ± 2.



$Y(t) = Z(t) + V(t)$

$$R_Z(\tau) = \begin{cases} 2^2 + 2^2\left(1 - \dfrac{|\tau|}{t_a}\right) = 4 + 4(100|\tau|), & |\tau| \le t_a = 0.01 \\ 4, & |\tau| > 0.01 \end{cases}$$

a) $$R_Y(\tau) = R_Z(\tau) + R_V(\tau) = \begin{cases} 2\delta(\tau) + 4 + 4(100|\tau|), & |\tau| \le 0.01 \\ 4, & |\tau| > 0.01 \end{cases}$$

b) $$S_Y(\omega) = \mathbf{F}\{R_Y(\tau)\} = 2 + 4 \times 2\pi\delta(\omega) + 4 \times \frac{1}{100} \times \text{sinc}^2\left(\frac{\omega}{100 \times 2\pi}\right)$$

$$= 2 + 8\pi\delta(\omega) + 0.04\text{sinc}^2\left(\frac{\omega}{200\pi}\right)$$

c) $R_{XY}(\tau) = R_{XZ}(\tau) + R_{XV}(\tau) = E\{X(t)Z(t+\tau)\} = E\{2X(t)\} + E\{X(t)Z_0(t+\tau)\}$, where $R_{XV}(\tau) = 0$. But, $E\{X(t)\} = 0$ also.

Therefore, $$R_{XY}(\tau) = R_{XZ_0}(\tau) = \begin{cases} 2 \times 8 \times (1 - 100|\tau|) = 16(1 - 100|\tau|), & |\tau| \le 0.01 \\ 0, & |\tau| > 0.01 \end{cases}$$