ECE 3075A
Random Signals

Lecture 5
Conditional Probability and Independence

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Combined Experiment Again

Clarification of notations

In die-throwing experiment, event $A = \{1, 3\} = \{1\} \cup \{3\}$ occurs if the outcome of the trial is either 1 or 3, and

$$\Pr(A) = \Pr\{1\} + \Pr\{3\} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

If we write B=$(1,3)$, a 2-tuple observation, then B is (and must be) an elementary outcome of a combined trial, consisting of 2 die-throwing, because a single die-throwing trial will never produce both 1 and 3 at the same time as the outcome; and if "the two trials of the combined experiment are independent",

$$\Pr(B) = \Pr\{1\} \times \Pr\{3\} = \frac{1}{6} \bullet \frac{1}{6} = \frac{1}{36}$$

with $(1,3)$ and $(3,1)$ treated as different outcomes.

Event & Conditional Event

- $\Pr(A)$ denotes the probability of event A.
- $\Pr(A \mid B)$ denotes the probability of event A, under the circumstance that event B has occurred; i.e., the probability of occurrence of an event conditioned by the occurrence of another event.

  let $A = \{1, 2, 3\}$ and $B = \{1, 4, 5, 6\}$ be two events associated with a die-throwing experiment.

➡ Given that B has occurred, meaning among the observation of the four outcomes – $1, 4, 5$, or 6, what is the probability that the occurrence of A is also observed? This is equivalent to asking about the probability of "1" when the observation space is B.

Conditional Probability

- If event B is assumed to have a non-zero probability, the conditional probability of A, given B, is

$$\Pr(A \mid B) = \frac{\Pr(A, B)}{\Pr(B)} = \frac{\Pr(A \cap B)}{\Pr(B)}, \qquad \Pr(B) > 0$$

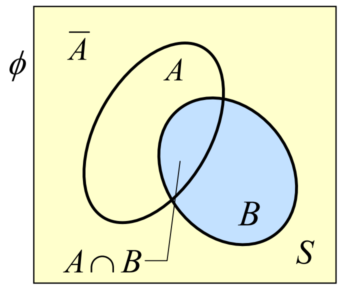
- We say joint event (A, B) occurs if the outcome of a trial satisfies the definition of both events, in other words, when any of the common elements of A and B appears as the outcome of the trial. That is,

$$\Pr(A, B) = \Pr(A \cap B)$$

In the previous example, $\Pr(A, B) = \Pr(A \cap B) = \Pr\{1\} = \frac{1}{6}$

Since $\Pr(B) = \frac{4}{6}$, $\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1/6}{4/6} = \frac{1}{4}$

Venn Diagram



$\Pr(B \mid B) = 1$

$\Pr(A \mid B)$ can be considered as the probability of A when B is the observation space.

$$\Pr(A \mid B) = \frac{\Pr(A,B)}{\Pr(B)} = \frac{\Pr(A \cap B)}{\Pr(B)},$$

provided $\Pr(B) > 0$

Joint Probability

$\Pr(A,B) = \Pr(A \cap B) = \Pr(\text{both events } A \text{ and } B \text{ are observed})$

$(A \cap B) \cup (\overline{A} \cap B) = S \cap B = B$, and $(A \cap B) \cap (\overline{A} \cap B) = \phi \;\Rightarrow\; \Pr(A,B) + \Pr(\overline{A},B) = \Pr(B)$

In general, if $A_i s$ **are mutually exclusive,** $\bigcup_i A_i = S$, $\sum_{All\ i} \Pr(A_i, B) = \Pr(B)$

Some Obvious Results

- If two events A and B are **mutually exclusive**,

  $A \cap B = \phi$ and $\Pr(A \cap B) = 0 \;\Rightarrow\; \Pr(A \mid B) = 0$

- If one event contains another:

$$A \subset B \;\Rightarrow\; \Pr(A \cap B) = \Pr(A) \;\Rightarrow\; \Pr(A \mid B) = \frac{\Pr(A)}{\Pr(B)} \ge \Pr(A)$$

$$B \subset A \;\Rightarrow\; \Pr(A \cap B) = \Pr(B) \;\Rightarrow\; \Pr(A \mid B) = \frac{\Pr(B)}{\Pr(B)} = 1$$

- In general, nothing can be asserted regarding the magnitude of conditional probability.

Conditional Probability and Axioms

- Non-negativity: $\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} \ge 0$
- "Sure" conditional event:

$$\Pr(S \mid B) = \frac{\Pr(S \cap B)}{\Pr(B)} = \frac{\Pr(B)}{\Pr(B)} = 1$$

- Exclusivity/Additivity

  Let C be another event such that $A \cap C = \phi$,

  $\Pr[(A \cup C) \cap B] = \Pr[(A \cap B) \cup (C \cap B)] = \Pr(A \cap B) + \Pr(C \cap B)$

  because $(A \cap B) \cap (C \cap B) = \phi$. Then

$$\Pr[(A \cup C) \mid B] = \frac{\Pr[(A \cup C) \cap B]}{\Pr(B)} = \frac{\Pr(A \cap B)}{\Pr(B)} + \frac{\Pr(C \cap B)}{\Pr(B)}$$

$$= \Pr(A \mid B) + \Pr(C \mid B)$$

Example of Conditional Probability

	Bin Number						
Ohms	1	2	3	4	5	6	Total
10	500	0	200	800	1200	1000	3700
100	300	400	600	200	800	0	2300
1000	200	600	200	600	0	1000	2600
Total	1000	1000	1000	1600	2000	2000	8600

What is the probability of drawing a 10 ohm resistor from any bin?
What is the probability of drawing a 100 ohm resistor from bin 3?
A 1000 ohm resistor was drawn. What is the probability that it came from bin 1?

Consider the experiment as one that produces (ohm, bin) as outcome.

Bayes Formula

Often used to find *a posteriori* probability

$$\Pr(x=i \mid y=j) = \frac{\Pr(y=j \mid x=i)\Pr(x=i)}{\Pr(y=j)}$$

More precisely,

$$\Pr(x=i \mid y=j) = \Pr(\text{symbol } i \text{ was sent, given that } j \text{ is received})$$

$$= \frac{\Pr(y=j \mid x=i)\Pr(x=i)}{\sum_{all\ i}\Pr(y=j \mid x=i)\Pr(x=i)}$$

$$= \frac{P_{ij}\Pr(x=i)}{\sum_{all\ i} P_{ij}\Pr(x=i)} = \frac{P_{ij}\Pr(x=i)}{\Pr(y=j)} =$$ the source portion of contributions that led to the reception of *j*

Independence

- If A and B are independent events of the same experiment (corresponding to a defined probability space), what can we say about them in conditional probabilities?

$$\Pr(A \cap B) = \Pr(A)\Pr(B)$$

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A)\Pr(B)}{\Pr(B)} = \Pr(A)$$

$$\Pr(B \mid A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{\Pr(A)\Pr(B)}{\Pr(A)} = \Pr(B)$$

That is, the probability assignment of event A has nothing to do with that of event B and *vice versa*.

Independence in Combined Experiment

- Consider two experiments: coin-tossing $S_1 = \{\text{H}, \text{T}\}$ and die-throwing $S_2 = \{1, 2, 3, 4, 5, 6\}$
- The cartesian product space has 12 elements:

$$S = S_1 \times S_2 = \{(\text{H},1), (\text{H},2), (\text{H},3), (\text{H},4), (\text{H},5), (\text{H},6), (\text{T},1), (\text{T},2), (\text{T},3), (\text{T},4), (\text{T},5), (\text{T},6)\}$$

- Examples of events

$$A_1 = \{\text{H}\} \subset S_1, \quad A_2 = \{1, 3, 5\} \subset S_2$$

$$\text{Then, } A = A_1 \times A_2 = \{(\text{H},1), (\text{H},3), (\text{H},5)\} \subset S_1 \times S_2$$

- Probability of combined events of **independent experiments**

$$\Pr(A) = \Pr(A_1 \times A_2) = \Pr(A_1)\Pr(A_2)$$

The probability (value) of event A in the cartesian product space is obtained as the product of the probabilities of A_1 and A_2. But, the original measures are related to different spaces, respectively.

- The notions of independent events and independent experiments are not exactly identical.