ECE 3075A
Random Signals

Lecture 4

Combined Experiments, Combinatorics

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Combined Experiments

- The outcomes of experiment 1 form an observation space $S_1 = \{\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n\}$ and those of experiment 2 form $S_2 = \{\beta_1, \beta_2, \beta_3, \cdots, \beta_m\}$.
- A **trial** in each experiment would produce an outcome in the respective observation space; the two respective outcomes can be taken together as if produced by a single trial of a **combined experiment.**
- The combined experiment then corresponds to a new observation space $S_1 \times S_2$ in which the elements are 2-tuples $(\alpha_1, \beta_1), (\alpha_1, \beta_2), (\alpha_1, \beta_3), \cdots, (\alpha_n, \beta_m)$
- $S_1 \times S_2$ is called a cartesian product space.
- A combined experiment may involve many experiments.

Example of Combined Experiment

- Consider two experiments: coin-tossing $S_1 = \{\text{H}, \text{T}\}$ and die- throwing $S_2 = \{1, 2, 3, 4, 5, 6\}$
- The cartesian product space has 12 elements:

$$S = S_1 \times S_2 = \{(\text{H},1), (\text{H},2), (\text{H},3), (\text{H},4), (\text{H},5), (\text{H},6), (\text{T},1), (\text{T},2), (\text{T},3), (\text{T},4), (\text{T},5), (\text{T},6)\}$$

- Examples of events

$$A_1 = \{\text{H}\} \subset S_1, \quad A_2 = \{1, 3, 5\} \subset S_2$$
$$\text{Then, } A = A_1 \times A_2 = \{(\text{H},1), (\text{H},3), (\text{H},5)\} \subset S_1 \times S_2$$

- Probability of combined events of **independent experiments** $\Pr(A) = \Pr(A_1 \times A_2) = \Pr(A_1)\Pr(A_2)$

The probability (value) of event A in the cartesian product space is obtained as the product of the probabilities of A_1 and A_2. But, the original measures are related to different spaces, respectively.

Bernoulli Trials

- Trials of the same experiment with outcomes combined to form a cartesian product space as in combined experiment. $S \Rightarrow S^n = S \times S \times \cdots \times S$
- Consider an event A.

$$\Pr(A) = p \text{ and } \Pr(\overline{A}) = q = 1 - p$$

- If we repeat the trial n times, assuming the trials are independent, then the outcome (a sequence), say,

$$(A, \overline{A}, \overline{A}, \cdots, A, \overline{A}) \in S^n$$

has probability

$$\Pr[(A, \overline{A}, \overline{A}, \cdots, A, \overline{A})] = \Pr(A)\Pr(\overline{A})\Pr(\overline{A})\cdots\Pr(A)\Pr(\overline{A}) = p^k q^{n-k}$$

assuming there are k and n-k occurrences of A and $\overline{A}$ in $(A, \overline{A}, \overline{A}, \cdots, A, \overline{A})$, respectively.

Cardinality of A Cartesian Space

- Let $\| S \| = (\text{number of elements in } S) = (\text{cardinality of } S) = I$
- Then $\| S^n \| = I^n$
- If all the elements in S^n are equiprobable, then

$$\Pr(B) = \frac{1}{I^n} \quad \text{for all } B \in S^n$$

Example:

Assume $p = q = 0.5$. Then,

$$\Pr(B) = \Pr[(A, \overline{A}, \overline{A}, \cdots, A, \overline{A})] = p^k q^{n-k} = \left(\frac{1}{2}\right)^n$$

If $p \neq q$, then the outcome sequences in S^n are not equiprobable.

Combinatorics

- Consider binary sequences of length n in which event A occurs exactly k times and $\overline{A}$ exactly n-k times.
- Among the entire 2^n possible sequences, $\frac{n!}{k!(n-k)!}$ of them are of this kind.
- Binomial coefficient ${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$ $\quad (p+q)^n = ?!$

Let event B_k={event A occurs k times in n trials}

$$\Pr(B_k) \equiv p_n(k) = \binom{n}{k} p^k q^{n-k}$$

DeMoivre-Laplace Theorem

$npq >> 1$ and

$|k\text{-}np| <\sim \sqrt{npq}$

$$p_n(k) = \binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi npq}} e^{-(k-np)^2/2npq}$$

Interpretation of DeMoivre-Laplace Theorem

- When the number of trials is large, i.e., $npq >> 1$ the mean approaches np and the variance approaches $np(1\text{-}p)=npq$
- Law of large numbers – asymptotic normality
- Deviation from normal distribution is small if k is in the vicinity of one standard deviation, i.e. $|k\text{-}np| <\sim \sqrt{npq} = \sigma$

$npq >> 1$ and $|k\text{-}np| <\sim \sqrt{npq}$

$$p_n(k) = \binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi npq}} e^{-(k-np)^2/2npq}$$

Application of Bernoulli Trial

- A fair die is rolled 6 times. Find the probability that "3" will show twice.

$$p = \Pr(A) = \Pr\{3\} = 1/6 \qquad \Pr(B_k) \equiv p_n(k) = \binom{n}{k} p^k q^{n-k}$$

$$\Pr(B_3) = \binom{6}{2}\left(\frac{1}{6}\right)^2\left(1-\frac{1}{6}\right)^{6-2} = 15 \times \frac{1}{6^2} \times \frac{5^4}{6^4} = \frac{1}{2}\left(\frac{5}{6}\right)^5$$

- A system containing n components is put into operation at t=0. The probability of any particular component will fail in the interval $(0,t)$ equals

$$p = \int_0^t \alpha(\tau)d\tau \quad \text{where } \alpha(t) \geq 0 \text{ and } \int_0^\infty \alpha(\tau)d\tau = 1$$

What is the probability that k of these components will fail prior to time t?

Ball-Drawing as Bernoulli Trials

- A box contains m white balls and n black balls. These balls are drawn at random, one at a time without replacement. What is the probability that (at least) a white ball is encountered by the kth draw?

Let $X_i = \{i \text{ black balls drawn followed by a white ball}\}$

$$W_k = \{\text{a white ball is encountered by the } k^{th} \text{ draw}\}$$
$$= X_0 \cup X_1 \cup X_2 \cup \cdots X_{k-1}$$

$X_0, X_1, X_2, \cdots, X_{k-1}$ are mutually exclusive $\Rightarrow \Pr(W_k) = \sum_{i=0}^{k-1} \Pr(X_i)$

$$\Pr(X_0) = \frac{m}{m+n} \qquad \Pr(X_1) = \frac{n}{m+n}\frac{m}{m+n-1} \qquad \Pr(X_{k-1}) = \frac{n(n-1)\cdots(n-k+1)m}{(m+n)(m+n-1)\cdots(m+n-k+1)}$$

$$\Pr(W_k) = \sum_{i=0}^{k-1} \Pr(X_i) = \frac{m}{m+n}\left[1 + \frac{n}{m+n-1} + \cdots \frac{n(n-1)\cdots(n-k+1)}{(m+n-1)\cdots(m+n-k+1)}\right]$$

$$\Pr(W_{n+1}) = ?$$

Ball-Drawing as Bernoulli Trials

- What is the observation space?

Let $X_i = \{i \text{ black balls drawn followed by a white ball}\}$

$\Longrightarrow$ Let $X_i = \{(\underbrace{\underbrace{\text{b}, \text{b}, \cdots, \text{b}}_{i}, \text{w}, \text{a}, \text{a}, \cdots, \text{a}}_{k})\}$

b = black
w = white
a = arbitrary

$W_k = \{\text{a white ball is encountered by the } k^{th} \text{ draw}\} = \{X_0, X_1, \cdots, X_{k-1}\}$

W_k is thus viewed as an event defined on the cartesian space S^k

$$\Pr(X_0) = \frac{m}{m+n} \bullet 1 \bullet 1 \cdots \bullet 1 \qquad \Pr(X_1) = \frac{n}{m+n}\frac{m}{m+n-1} \bullet 1 \bullet 1 \cdots \bullet 1$$

$$\Pr(X_{k-1}) = \frac{n(n-1)\cdots(n-k+1)m}{(m+n)(m+n-1)\cdots(m+n-k+1)}$$

$$\Pr(W_k) = \sum_{i=0}^{k-1} \Pr(X_i) = \frac{m}{m+n}\left[1 + \frac{n}{m+n-1} + \cdots \frac{n(n-1)\cdots(n-k+1)}{(m+n-1)\cdots(m+n-k+1)}\right]$$