ECE 3075A
Random Signals

Lecture 35
Crosscorrelation, System Analysis in Time & Frequency Domain

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Input-Output Crosscorrelation

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = E\left[X(t)\int_0^\infty X(t+\tau-\lambda)h(\lambda)d\lambda\right]$$

$$= \int_0^\infty E[X(t)X(t+\tau-\lambda)]h(\lambda)d\lambda$$

$$= \int_0^\infty R_X(\tau-\lambda)h(\lambda)d\lambda$$

The crosscorrelation function is the convolution of the input auto-correlation and the impulse response of the system!!

$$R_{YX}(\tau) = E[X(t+\tau)Y(t)] = \int_0^\infty E[X(t+\tau)X(t-\lambda)]h(\lambda)d\lambda$$

$$= \int_0^\infty R_X(\tau+\lambda)h(\lambda)d\lambda$$

Note: $R_{XY}(-\tau) = \int_0^\infty R_X(-\tau-\lambda)h(\lambda)d\lambda = \int_0^\infty R_X(\tau+\lambda)h(\lambda)d\lambda = R_{YX}(\tau)$

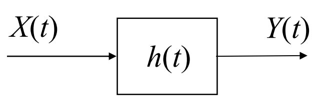
Crosscorrelation is useful in finding the impulse response of a linear system.

Crosscorrelation with White Noise Input

Consider a white noise input signal whose autocorrelation is

$$R_X(\tau) = \frac{N_0}{2}\delta(\tau)$$

$$Y(t) = \int_0^\infty X(t-\lambda)h(\lambda)d\lambda$$



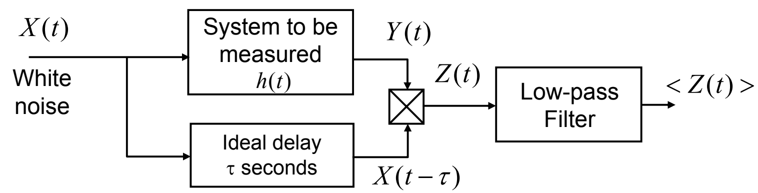
$$R_{XY}(\tau) = \int_0^\infty R_X(\tau-\lambda)h(\lambda)d\lambda$$

$$= \int_0^\infty \frac{N_0}{2}\delta(\tau-\lambda)h(\lambda)d\lambda = \begin{cases} \frac{N_0}{2}h(\tau), & \tau \ge 0 \\ 0, & \tau < 0 \end{cases}$$

Similarly, $R_{YX}(\tau) = \int_0^\infty \frac{N_0}{2}\delta(\tau+\lambda)h(\lambda)d\lambda = \begin{cases} 0, & \tau > 0 \\ \frac{N_0}{2}h(-\tau), & \tau < 0 \end{cases}$

The crosscorrelation function is proportional to the impulse response!

Method for Measuring Impulse Response



$$Y(t) = \int_0^\infty X(t-\lambda)h(\lambda)d\lambda$$

$Z(t) = X(t-\tau)Y(t)$ and $E[Z(t)] = E[X(t-\tau)Y(t)] = R_{XY}(\tau)$

A properly designed low-pass filter will be able to produce a time average of $Z(t)$, i.e., $< Z(t) >$ which is equal to $E[Z(t)]$ for an ergodic process.

$$< Z(t) > \cong R_{XY}(\tau) = \frac{N_0}{2}h(\tau) \quad \text{for } \tau \ge 0$$

An Alternative Expression of Output Autocorrelation

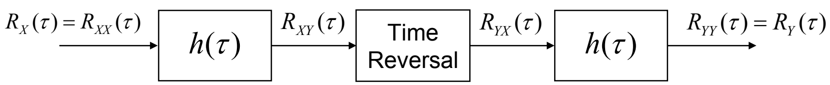
$$R_Y(\tau) = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1 - \tau) h(\lambda_1) h(\lambda_2) d\lambda_2$$

$$= \int_0^\infty \left\{ \int_0^\infty R_X(\lambda_2 - \tau - \lambda_1) h(\lambda_1) d\lambda_1 \right\} h(\lambda_2) d\lambda_2 = \int_0^\infty R_{XY}(\lambda_2 - \tau) h(\lambda_2) d\lambda_2$$

In the above, $R_{XY}(\lambda_2 - \tau) = \int_0^\infty R_X(\lambda_2 - \tau - \lambda_1) h(\lambda_1) d\lambda_1$

Or in general form, $R_{XY}(\tau) = \int_0^\infty R_X(\tau - \lambda) h(\lambda) d\lambda$

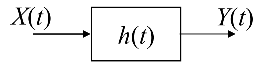
And $R_Y(\tau) = R_{YY}(\tau) = \int_0^\infty R_{XY}(\lambda - \tau) h(\lambda) d\lambda = \int_0^\infty R_{YX}(\tau - \lambda) h(\lambda) d\lambda$

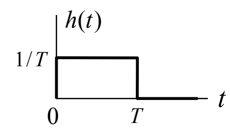


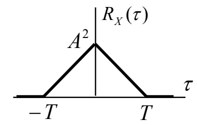
Calculation of output autocorrelation via convolution.

Time Domain Analysis - Example

Consider a finite-time integrator which has an impulse response







Input autocorrelation

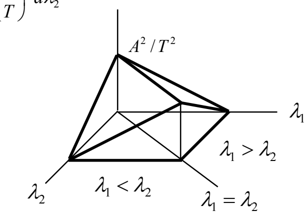
$$h(t) = \frac{1}{T}[u(t) - u(t-T)]$$

$$\overline{Y} = \overline{X}\int_0^T \frac{1}{T} dt = \overline{X}$$

$$\overline{Y^2} = \int_0^T d\lambda_1 \int_0^T R_X(\lambda_2 - \lambda_1) \left(\frac{1}{T}\right)^2 d\lambda_2 = \int_0^T d\lambda_1 \int_0^T A^2 \left[1 - \frac{|\lambda_2 - \lambda_1|}{T}\right] \left(\frac{1}{T}\right)^2 d\lambda_2$$

$$= \frac{A^2}{T^2} \left\{ \int_0^T \int_0^{\lambda_2} \left[1 - \frac{\lambda_2 - \lambda_1}{T}\right] d\lambda_1 d\lambda_2 + \int_0^T \int_0^{\lambda_1} \left[1 - \frac{\lambda_1 - \lambda_2}{T}\right] d\lambda_2 d\lambda_1 \right\}$$

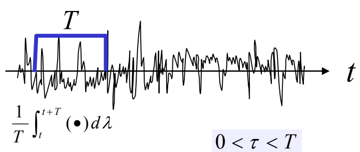
$$= \frac{A^2}{T^2}\left(\frac{T^2}{3} + \frac{T^2}{3}\right) = \frac{2A^2}{3}$$



In a similar manner, one can find the autocorrelation function of the output.

Finite Time Average of White Noise

The input is zero mean white noise, $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$



$$\overline{Y} = \overline{X}\int_0^T \frac{1}{T} dt = \overline{X} = 0$$

$$R_Y(\tau) = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1 - \tau) h(\lambda_1) h(\lambda_2) d\lambda_2$$

$$= \frac{N_0}{2} \int_0^T d\lambda_1 \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) \frac{1}{T}\frac{1}{T} d\lambda_2$$

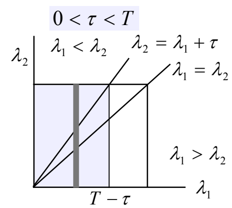
$$\int_0^T \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) d\lambda_1 d\lambda_2$$

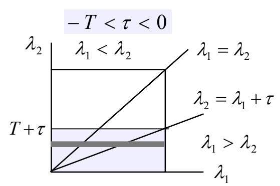
$$= \begin{cases} \int_0^{T-\tau} \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) d\lambda_2 d\lambda_1, & 0 < \tau < T \\ \int_0^{T+\tau} \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) d\lambda_1 d\lambda_2, & -T < \tau < 0 \end{cases}$$

$$= T - |\tau| \quad {\scriptstyle |\tau| < T}$$

Therefore, $R_Y(\tau) = \frac{N_0}{2T}\left(1 - \frac{|\tau|}{T}\right)$

And, $\overline{Y^2} = R_Y(\tau = 0) = \frac{N_0}{2T}$





Finite Time Average of White Noise

With zero mean white noise, $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$

$$R_Y(\tau) = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2 - \lambda_1 - \tau) h(\lambda_1) h(\lambda_2) d\lambda_2 = \frac{N_0}{2T^2} \int_0^T d\lambda_1 \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) d\lambda_2$$

$$\int_0^T \int_0^T \delta(\lambda_2 - \lambda_1 - \tau) d\lambda_1 d\lambda_2 = T - |\tau| \quad {\scriptstyle |\tau| < T} \quad \Longrightarrow \quad R_Y(\tau) = \frac{N_0}{2T}\left(1 - \frac{|\tau|}{T}\right)$$

What is the corresponding transfer function?

Example 8-6.1 White noise having a two-sided spectral density of 0.80 is applied to the input of a finite-time integrator having an impulse response of $h(t) = \frac{1}{4}[u(t) - u(t-4)]$

Find the value of the autocorrelation function of the output at a) $\tau = 0$; b) $\tau = 1$; c) $\tau = 2$.

$T = 4$ and $(N_0/2) = 0.8$ $\quad R_Y(\tau) = \frac{0.8}{4}\left(1 - \frac{|\tau|}{4}\right) = \frac{1}{5}\left(1 - \frac{|\tau|}{4}\right)$

$R_Y(0) = \frac{1}{5}\left(1 - \frac{|0|}{4}\right) = 0.2$ $\quad R_Y(1) = \frac{1}{5}\left(1 - \frac{1}{4}\right) = \frac{3}{20} = 0.15$ $\quad R_Y(2) = \frac{1}{5}\left(1 - \frac{2}{4}\right) = 0.1$

Analysis of "Signal Plus Noise"

Consider a signal/process $X(t) = A + V(t)$ where the noise $V(t)$ has an autocorrelation function of $R_V(\tau) = 10e^{-1000|\tau|}$.
An RC circuit is being used to filter out the noise and the requirement is to measure A with an error of 1% when A is on the order of 1. Determine the RC time constant.

For the noise: $R_V(\tau) = \frac{\beta S_0}{2} e^{-\beta|\tau|} = 10e^{-1000|\tau|}$ $\beta = 1000,\ S_0 = 0.02$
A wideband noise

$S_V(\omega) = \int_{-\infty}^{\infty} R_V(\tau) e^{-j\omega\tau} d\tau$ and $S_V(0) = \int_{-\infty}^{\infty} R_V(\tau) d\tau = 2\int_0^{\infty} 10e^{-1000\tau} d\tau = 0.02$

around $\omega = 0$, $S_V(\omega) \approx S_V(0) = 0.02$

$Y(t) = A + U(t)$ where $U(t)$ is the filtered version of $V(t)$

$$\overline{U^2} = R_U(0) \approx \frac{bN_0}{4} = \frac{bS_V(0)}{2} = 0.01b, \quad \text{or} \quad \sqrt{\overline{U^2}} \approx 0.1\sqrt{b}$$

The requirement dictates that $\sqrt{\overline{U^2}} \approx 0.1\sqrt{b} \le 1\% = 0.01 \quad \Rightarrow \quad b \le 0.01$

Since $b = 1/RC \quad \Rightarrow \quad RC \ge 100$

Example 8-6.2

Consider a process of a dc signal plus noise $X(t) = A + V(t)$ where $V(t)$ has an autocorrelation function of $R_V(\tau) = 1 - \frac{|\tau|}{0.02}$ for $|\tau| \le 0.02$. A finite time integrator is used to estimate the value of A with the expectation that the rms error is less than 0.01. If the impulse response of the integrator is $h(t) = \frac{1}{T}[u(t) - u(t-T)]$ find the value of T to accomplish this.

$$S_V(0) = \int_{-\infty}^{\infty} R_V(\tau) d\tau = 2\int_0^{0.02} \left(1 - \frac{\tau}{0.02}\right) d\tau = 2 \times 0.01 = 0.02$$

around $\omega = 0$, $S_V(\omega) \approx S_V(0) = 0.02$

$Y(t) = A + U(t)$ where $U(t)$ is the filtered version of $V(t)$

Since the signal is a constant and the expected error is small, it is reasonable to assume that within the passband, the noise power is constant (equivalent to white noise).

$$R_U(\tau) = \frac{S_V(0)}{T}\left(1 - \frac{|\tau|}{T}\right) \text{ and } \overline{U^2} = R_U(0) = \frac{S_V(0)}{T} = \frac{0.02}{T}$$

We require that $\overline{U^2} = \frac{0.02}{T} \le (0.01)^2$. Therefore, $T \ge 200$.