ECE 3075A
Random Signals

Lecture 30

Spectral Density and Autocorrelation Functions

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Properties of Spectral Density

- Spectral density is a real, non-negative and even function of frequency (ω or $f[req]$).
- Since it is an even function of the frequency, a rational spectral density of the form

$$S_X(\omega) = \frac{S_0(\omega^{2n} + a_{2n-2}\omega^{2n-2} + \cdots + a_2\omega^2 + a_0)}{\omega^{2m} + b_{2m-2}\omega^{2m-2} + \cdots + b_2\omega^2 + b_0}$$

  contains only even powers of ω.
- White noise is a process whose spectral density is a constant over the entire frequency range, i.e.,

$$S_X(\omega) = S_0 \quad \text{for all } \omega$$

  (just like a "white" light that contains all colored lights with equal intensity). When the marginal distribution is Gaussian, we call it a **Gaussian white noise**.

Spectral Density of Constant or Periodic Signals

Consider a process $X(t) = A + B\cos(2\pi f_0 t + \Theta)$

where A, B and f_0 are constant and Θ is a random variable uniformly distributed over (0,2π).

Let $X_T(t)$ be a truncated version of $X(t)$ over $(-T, T)$.

$$S_X(\omega) = \lim_{T\to\infty} \frac{E[|F_{X_T}(\omega)|^2]}{2T}$$

$$F_{X_T}(f) = \int_{-T}^{T} [A + B\cos(2\pi f_0 + \Theta)] e^{-j2\pi ft} dt$$

$$\mathbf{F}[X(t)] = \int_{-\infty}^{\infty} [A + B\cos(2\pi f_0 + \Theta)] e^{-j2\pi ft} dt$$

$$= A\delta(f) + (B/2)\left[\delta(f+f_0)e^{-j\Theta} + \delta(f-f_0)e^{j\Theta}\right]$$

$$F_{X_T}(f) = 2T\text{sinc}(2Tf) * \mathbf{F}[X(t)]$$

$$= 2AT\text{sinc}(2Tf) + BT\{\text{sinc}[2(f+f_0)T]e^{-j\Theta} + \text{sinc}[2(f-f_0)T]e^{j\Theta}\}$$

$$|F_{X_T}(f)|^2 = F_{X_T}(f)F^*_{X_T}(f)$$

$$= (2AT)^2\text{sinc}^2(2Tf) + (BT)^2\left\{\text{sinc}^2[2(f+f_0)T] + \text{sinc}^2[2(f-f_0)T]\right\}$$

$$+ C(f)e^{-j\Theta} + C(-f)e^{j\Theta} + D(f)e^{-j2\Theta} + D(-f)e^{j2\Theta}$$

Spectral Density of Constant or Periodic Signals

$$E\left[|F_{X_T}(f)|^2\right]$$

$$= (2AT)^2\text{sinc}^2(2Tf) + (BT)^2\left\{\text{sinc}^2[2(f+f_0)T] + \text{sinc}^2[2(f-f_0)T]\right\}$$

because $E_\Theta[G(f)e^{-j\Theta}] = 0$ and $E_\Theta[H(f)e^{-j2\Theta}] = 0$ for Θ uniformly distributed in $(0, 2\pi)$

$$S_X(f) = \lim_{T\to\infty} \frac{E\left[|F_{X_T}(f)|^2\right]}{2T}$$

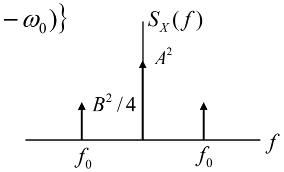
$$= \lim_{T\to\infty}\left\{A^2 2T\text{sinc}^2(2Tf) + (B^2/4)\left(2T\text{sinc}^2[2(f+f_0)T] + 2T\text{sinc}^2[2(f-f_0)T]\right)\right\}$$

But, $\lim_{T\to\infty} 2T\text{sinc}^2(2Tf) = \delta(f)$

Therefore, $S_X(f) = A^2\delta(f) + (B^2/4)\{\delta(f+f_0) + \delta(f-f_0)\}$

And, $S_X(\omega) = 2\pi A^2\delta(\omega) + (\pi B^2/2)\{\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\}$

The spectral density thus consists of three spikes (delta functions) at DC (with height A^2) and at $\pm f_0$ (with height $B^2/4$), respectively.



Mean-Square Value and Total Power

$X(t) = A + B\cos(\omega_0 t + \Theta)$ Θ uniformly distributed in $(0,2\pi)$

$S_X(\omega) = 2\pi A^2\delta(\omega) + (\pi B^2/2)\{\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\}$

Total power:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{2\pi A^2\delta(\omega) + (\pi B^2/2)[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]\}d\omega$$

$$= A^2 + \frac{B^2}{4} + \frac{B^2}{4} = A^2 + \frac{B^2}{2}$$

Mean-Square Value of the process

$$E_\Theta[X^2(t)] = E_\Theta\{[A + B\cos(\omega_0 t+\Theta)]^2\} = E_\Theta\{A^2 + 2AB\cos(\omega_0 t+\Theta) + B^2\cos^2(\omega_0 t+\Theta)\}$$

$$= A^2 + E_\Theta\{2AB\cos(\omega_0 t+\Theta)\} + B^2 E_\Theta[\cos^2(\omega_0 t+\Theta)]$$

$E_\Theta\{2AB\cos(\omega_0 t+\Theta)\} = 0$ for $\Theta \sim U(0,2\pi)$

$$E_\Theta\{\cos^2(\omega_0 t+\Theta)\} = E_\Theta\left\{\frac{\cos(2\omega_0 t + 2\Theta)+1}{2}\right\} = \frac{1}{2}$$

Therefore, $E_\Theta[X^2(t)] = \overline{X^2} = A^2 + \frac{B^2}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega$

Example 7-3.1

A stationary random process has a spectral density of the form

$$S_X(f) = 4\delta(f) + 18\delta(f+8) + 18\delta(f-8)$$

a. Find the discrete frequencies present;
b. Find the mean value of the process;
c. Find the variance of the process.

$S_X(f) = A^2\delta(f) + (B^2/4)\{\delta(f+f_0) + \delta(f-f_0)\}$

Frequencies present are 0, +8 and –8 Hz respectively.

The mean value of the process corresponds to the amplitude of the DC (i.e., zero-frequency) component which has a power spectral density of 4. Therefore, the mean is ±2.

The mean-square value is 4+18+18=40

Thus, the variance is $\sigma^2 = \overline{X^2} - (\overline{X})^2 = 40 - 4 = 36$

Time Derivative of A Random Process

Let $\dot{X}(t) = dX(t)/dt$

$$X(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_X(\omega)e^{j\omega t}d\omega$$

$$\dot{X}(t) = \frac{dX(t)}{dt} = \frac{d}{dt}\frac{1}{2\pi}\int_{-\infty}^{\infty} F_X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega F_X(\omega)e^{j\omega t}d\omega$$

$$\mathbf{F}[\dot{X}(t)] = j\omega F_X(\omega)$$

$$S_{\dot{X}}(\omega) = \lim_{T\to\infty}\frac{E[|j\omega F_{X_T}(\omega)(-j\omega)F_{X_T}(-\omega)|]}{2T} = \omega^2 \lim_{T\to\infty}\frac{E[|F_{X_T}(\omega)|^2]}{2T} = \omega^2 S_X(\omega)$$

In case $S_X(\omega)$ does not drop off more rapidly than $1/\omega^2$ as $\omega \to \infty$, then the mean - square value of the derivative process will become infinite. The process is then said to be non - differentiable.