ECE 3075A
Random Signals

Lecture 3
Event Counting and Combined Experiment

School of Electrical and Computer Engineering
Georgia Institute of Technology
Fall, 2003

Syllabus

- Lecturer: Professor B.H. Juang
- Lecture time: MWF 1505:1555 @ Van Leer C341
- Office hours: MF 1400-1455 @ Bunger Henry 310
- GTA: Ilker Capoglu
  Office hours: TBA
  Tutorial Lab: VL C448
- Text: Cooper & McGillem, *Probabilistic Methods of Signal & System Analysis*, 3rd edition
- Homework, lecture note and other information:
  http://users.ece.gatech.edu/~juang

De Morgan's Law

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

In a set identity, if we replace all sets by their complements, all unions by intersections, all intersections by unions, the identity is preserved. For example,

$$\overline{A \cap (B \cup C)} = \overline{A} \cup \overline{B \cup C} = \overline{A} \cup (\overline{B} \cap \overline{C})$$

$$\overline{(A \cap B) \cup (A \cap C)} = \overline{A \cap B} \cap \overline{A \cap C} = (\overline{A} \cup \overline{B}) \cap (\overline{A} \cup \overline{C})$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$\overline{A} \cup (\overline{B} \cap \overline{C}) = (\overline{A} \cup \overline{B}) \cap (\overline{A} \cup \overline{C})$$

Axiomatic Approach

- Built upon set theory
- A **Probability Space** consists of three components: the observation or sample space, S, set algebra, and the probability measure, Pr(), which assigns a probability to each and every event.
- The **Observation Space** is a space consisting of all the outcomes of a random experiment.
- **Events** are subsets of the observation space.
- Pr() must satisfy **three axioms**:

Non-negativity:	$\Pr(A) \geq 0$
Sure event:	$\Pr(S) = 1$
Exclusivity:	If $A \cap B = \phi$, then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$

Corollaries from the Axioms

$S \cap \phi = \phi$ and $S \cup \phi = S \quad \Rightarrow$

$$\Pr(S \cup \phi) = \Pr(S) = \Pr(S) + \Pr(\phi) \quad \Rightarrow \quad \Pr(\phi) = 0$$

$A \cap \overline{A} = \phi$ and $A \cup \overline{A} = S \quad \Rightarrow \quad \Pr(S) = \Pr(A \cup \overline{A}) = \Pr(A) + \Pr(\overline{A}) = 1$

$$\Rightarrow \quad \Pr(A) = 1 - \Pr(\overline{A}) \le 1$$

$A \cup B = A \cup (\overline{A} \cap B)$ and $A \cap (\overline{A} \cap B) = \phi$

$\Rightarrow \quad \Pr(A \cup B) = \Pr[A \cup (\overline{A} \cap B)] = \Pr(A) + \Pr(\overline{A} \cap B)$

$B = (A \cap B) \cup (\overline{A} \cap B)$ and $(A \cap B) \cap (\overline{A} \cap B) = \phi \quad \Rightarrow$

$\Pr(B) = \Pr(A \cap B) + \Pr(\overline{A} \cap B)$ or $\Pr(\overline{A} \cap B) = \Pr(B) - \Pr(A \cap B)$

$$\Pr(A \cup B) = \Pr(A) + \Pr(\overline{A} \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \le \Pr(A) + \Pr(B)$$

Important Corollaries

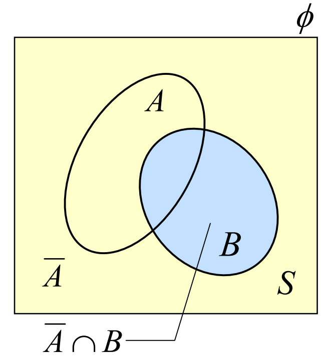
- 0-measure events:

  $\Pr(\phi) = 0$

  Theoretically, there could be other 0-measure events than the empty set – the axioms allow that. But in the relative frequency approach, they are usually either not treated or treated differently as "unseen events".

- Complementary events:

  $\Pr(A) = 1 - \Pr(\overline{A}) \le 1$



- $\Pr(A \cup B) = \Pr(A) + \Pr(\overline{A} \cap B)$
- $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \le \Pr(A) + \Pr(B)$

Example

- Die-throwing experiment

  $S = \{1, 2, 3, 4, 5, 6\}$

  Let $\Pr(1) = p_1, \Pr(2) = p_2, \cdots, \Pr(6) = p_6$

  For an unbiased die, we expect $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6}$

  Let $A = \{1, 4\} = \{1\} \cup \{4\}$, then $\Pr(A) = \Pr(1) + \Pr(4) = p_1 + p_4$

  Let $B = \{1, 5, 6\} = \{1\} \cup \{5\} \cup \{6\}$, then $\Pr(B) = p_1 + p_5 + p_6$

  $C = A \cup B = \{1, 4, 5, 6\} = A \cup (\overline{A} \cap B)$

  $\Pr(C) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

  $\quad = p_1 + p_4 + p_1 + p_5 + p_6 - p_1 = p_1 + p_4 + p_5 + p_6$

Independent Events

- Events in a random experiment are called independent if $\Pr(A \cap B) = \Pr(A)\Pr(B)$
- If events A and B are independent, then the events $\overline{A}$ and B and the events $\overline{A}$ and $\overline{B}$ are also independent.
- The concept of independent events may be at times confusing. Example: in die-throwing experiment,

  $S = \{1, 2, 3, 4, 5, 6\} \qquad \Pr(1) = p_1, \Pr(2) = p_2, \cdots, \Pr(6) = p_6$

  assume $p_1 = p_3 = p_5 = p_2 = p_4 = p_6 = (1/6)$

  $\Pr(even) = p_2 + p_4 + p_6 = 0.5 \qquad \Pr(odd) = p_1 + p_3 + p_5 = 0.5$

  $\Pr(\{odd\} \cap \{even\}) = \Pr(\phi) = 0 \ne \Pr(odd)\Pr(even) = 0.25$

  i.e., $\{odd\}$ and $\{even\}$ are mutually exclusive but not independent

Combined Experiments

- The outcomes of experiment 1 form an observation space $S_1 = \{\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n\}$ and those of experiment 2 form $S_2 = \{\beta_1, \beta_2, \beta_3, \cdots, \beta_m\}$.
- A **trial** in each experiment produces an outcome in the respective observation space; the two respective outcomes can be taken together as if produced by a single trial of a **combined experiment.**
- The combined experiment then corresponds to a new observation space $S_1 \times S_2$ in which the elements are 2-tuples $(\alpha_1, \beta_1), (\alpha_1, \beta_2), (\alpha_1, \beta_3), \cdots, (\alpha_n, \beta_m)$
- $S_1 \times S_2$ is called a cartesian product space.
- A combined experiment may involve many experiments.

Example of Combined Experiment

- Consider two experiments: coin-tossing $S_1 = \{\text{H}, \text{T}\}$ and die-throwing $S_2 = \{1, 2, 3, 4, 5, 6\}$
- The cartesian product space has 12 elements:

$$S = S_1 \times S_2 = \{(\text{H},1), (\text{H},2), (\text{H},3), (\text{H},4), (\text{H},5), (\text{H},6), (\text{T},1), (\text{T},2), (\text{T},3), (\text{T},4), (\text{T},5), (\text{T},6)\}$$

- Examples of events

$$A_1 = \{\text{H}\} \subset S_1, \quad A_2 = \{1, 3, 5\} \subset S_2$$

$$\text{Then, } A = A_1 \times A_2 = \{(\text{H},1), (\text{H},3), (\text{H},5)\} \subset S_1 \times S_2$$

- Probability of combined events of **independent experiments**

$$\Pr(A) = \Pr(A_1 \times A_2) = \Pr(A_1)\Pr(A_2)$$

The probability (value) of event A in the cartesian product space is obtained as the product of the probabilities of A_1 and A_2. But, the individual measures are originally related to different spaces.

Revisit to Relative Frequency of Occurrences

- Use the die-throwing experiment as example.
- S has 6 possible outcomes.
- Let $N_i, i = 1, 2, \cdots, 6$, be the number of occurrences of these possible outcomes after $N = \sum_{i=1}^{6} N_i$ trials.
- The relative frequency of occurrences are defined as

$$r_i = N_i / N, \quad i = 1, 2, \cdots, 6$$

- The relative frequency of occurrences can be used as an **empirical estimate** of probability, i.e., $p_i \cong r_i$
- If $N \to \infty$ and the die is unbiased or fair, we expect

$$r_1 = r_2 = r_3 = r_4 = r_5 = r_6 = \frac{1}{6}$$

Events of Equally Probable Outcomes

- Therefore, **if the outcomes are equally probable**, they can serve as the counting unit when calculating the probability of an event.

$$A = \{\text{odd number}\} = \{1, 3, 5\}, \quad \Pr(A) = 3 \times \left(\frac{1}{6}\right) = \frac{1}{2}$$

This event accounts for three out of the six possible outcomes, and thus has probability one half.

$$B = \{\text{number less than } 3\} = \{1, 2\}, \quad \Pr(A) = 2 \times \left(\frac{1}{6}\right) = \frac{1}{3}$$

This event accounts for two out of the six possible outcomes, and thus has probability one third.

This is true only if the elementary outcomes are equally probable.
This is also the basis of the classical definition of probability.