ECE 3075A
Random Signals

Lecture 25
Autocorrelation Functions & Their Properties

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Correlation Functions

- Correlation between two random variables is the expected value of the product of the two random variables. It measures how "coherently" the two random variables behave. If they behave coherently (e.g., when one r.v. is observed to have a high value, the other is likely to have a high value as well), the correlation is high.

Correlation: $R_{XY} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dxdy$

Correlation Coefficient: $\rho_{XY} = E\left[\frac{(X-\bar{X})}{\sigma_X}\frac{(Y-\bar{Y})}{\sigma_Y}\right]$

- A correlation function can be defined in a similar manner between two random processes to measure how "coherently" the two random processes behave. These two processes can refer to the same random process – when this is the case, it is called the autocorrelation function.

Correlation Functions

- The correlation function between two different random processes is called **cross-correlation**, defined as

$$R_{XY}(t_1,t_2) = E[X(t_1)Y(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 y_2 f_{X_1,Y_2}(x_1,y_2)\,dx_1 dy_2$$

where we use the notation $X(t_1) = X_1$ and $Y(t_2) = Y_2$ of which x_1 and y_2 are a realization, respectively.

- The **autocorrelation function** can be defined in a similar manner, with Y in the above replaced by X:

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X_1,X_2}(x_1,x_2)\,dx_1 dx_2$$

Obviously,

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = E[X(t_2)X(t_1)] = R_X(t_2,t_1)$$

$$R_X(t,t) = E[X(t)X(t)] = E[X^2(t)] \ge 0$$

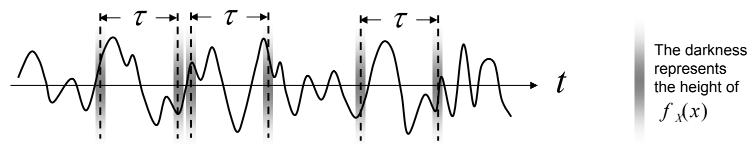
Wide Sense Stationary Processes

$$R_X(t_1,t_2) = R_X(t_1+T,t_2+T) = E[X(t_1+T)X(t_2+T)]$$

Set $T = -t_1 \quad \Rightarrow \quad R_X(t_1,t_2) = R_X(t_1-T,t_2-T) = R_X(0,t_2-t_1)$

For a wide sense stationary process, the autocorrelation function does not depend on the absolute time origin. The first argument 0 is thus arbitrary and the autocorrelation is a function of only the time difference, $t_2 - t_1 = \tau$.

$$R_X(\tau) = R_X(0,t_2-t_1) = E[X(t_1)X(t_1+\tau)] = E[X(t)X(t+\tau)]$$



If we look at any two time instances of a wide sense stationary process, their correlation is only a function of their time difference, no matter where they are. In subsequent discussions, wide sense stationarity is always assumed.

Time Autocorrelation Function

$$\mathbf{R}_X(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)\,dt = \langle x(t)x(t+\tau)\rangle$$

For an ergodic process, $\mathbf{R}_X(\tau) = R_X(\tau)$

$R_X(0) = E[X(t)X(t)] = E[X^2(t)] =$ mean-square value of the process.

$$\mathbf{R}_X(0) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^2(t)\,dt = \langle x^2(t)\rangle$$

$R_X(0)$ is used as the average or the expected power of a random process.

Example:

In many cases, the noise that gets into a signal is considered I.I.d. with zero mean and variance σ^2. For such processes,

$$R_V(\tau) = E[V(t)V(t+\tau)] = \begin{cases} \sigma^2, & \tau = 0 \\ 0, & \tau \neq 0 \end{cases}$$

$R_V(0)$ is considered the power of noise.

Interpretation of Autocorrelation

Let $X(t)$ be a zero-mean stationary random process. Form a new random process $Y(t)$ according to $Y(t) = X(t) - \rho X(t+\tau)$.
If the variation in Y is small, we can use $X(t)$ to "predict" $X(t+\tau)$; obtain an early estimate of $X(t+\tau)$ by just dividing the value of $X(t)$ by ρ. But how do we choose ρ such that variation in Y is small?

⟹ We choose ρ to minimize $E[Y^2(t)]$.

$$E[Y^2(t)] = E\{[X(t) - \rho X(t+\tau)]^2\} = E\{X^2(t) - 2\rho X(t)X(t+\tau) + \rho^2 X^2(t+\tau)\}$$

That is, $\sigma_Y^2 = \sigma_X^2 - 2\rho R_X(\tau) + \rho^2\sigma_X^2$

$$\frac{d\sigma_Y^2}{d\rho} = -2R_X(\tau) + 2\rho\sigma_X^2 = 0 \;\Rightarrow\; \rho = \frac{R_X(\tau)}{\sigma_X^2}$$ which is the correlation coefficient

Therefore, for simple prediction of $X(t+\tau)$, we use the value $X(t)$ and divide it by the correlation coefficient ρ between $X(t)$ and $X(t+\tau)$. For better prediction, higher order is often needed.

Example 6-1.1

A random process has sample functions of the form on the right:

$$X(t) = \begin{cases} A, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Where A is a random variable uniformly distributed from 0 to 10. Using the basic definition of the autocorrelation function as given by eq. 6-1, find the autocorrelation of the process.

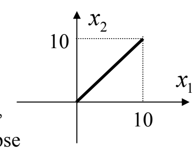
$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{XX}(x_1,x_2)\,dx_1 dx_2$$

$X(t)$ is a deterministic function because once A is realized, the time function is known.

If $0 \le t_1, t_2 \le 1$,

$$R_X(t_1,t_2) = \int_0^{10} x^2 \frac{1}{10}\,dx = \frac{1}{10}\frac{x^3}{3}\Big|_0^{10} = 33.3$$

If $0 \le t_1, t_2 \le 1$, $f_{X_1,X_2}(x_1,x_2) = 0.1$ on the line $x_1 = x_2$; 0 elsewhere. The (x_1, x_2) corrdinates collapse to a line.



If t_1 and t_2 are not in (0,1) simultaneously,

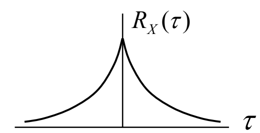
$R_X(t_1,t_2) = 0$, because at least one $X = 0$, making the product and the integral zero.

Example 6-1.2

Define $Z(t) = X(t) + X(t+\tau_1)$ where $X(t)$ is a sample function from a stationary process whose autocorrelation function is

$$R_X(\tau) = \exp(-\tau^2)$$

Write an expression for the autocorrelation function of the random process $Z(t)$.



$$\begin{aligned} R_Z(\tau) &= E[Z(t)Z(t+\tau)] \\ &= E\{[X(t) + X(t+\tau_1)][X(t+\tau) + X(t+\tau+\tau_1)]\} \\ &= E[X(t)X(t+\tau) + X(t+\tau_1)X(t+\tau) \\ &\quad + X(t)X(t+\tau+\tau_1) + X(t+\tau_1)X(t+\tau+\tau_1)) \\ &= R_X(\tau) + R_X(\tau-\tau_1) + R_X(\tau+\tau_1) + R_X(\tau) \\ &= 2\exp(-\tau^2) + \exp[-(\tau-\tau_1)^2] + \exp[-(\tau+\tau_1)^2] \end{aligned}$$

Properties of Autocorrelation Functions

- $R_X(0) = \overline{X^2} \geq 0$
- Symmetry: $R_X(\tau) = R_X(-\tau)$

$$R_X(\tau) = E[X(t)X(t+\tau) = E[X(t-\tau)X(t)] = R_X(-\tau)$$

- $|R_X(\tau)| \leq R_X(0)$

$$E[(X_1 \pm X_2)^2] = E[X_1^2 \pm 2X_1X_2 + X_2^2] \geq 0$$

$$E[X_1^2 + X_2{}^2] = 2R_X(0) \geq |E[2X_1X_2]| = 2|R_X(\tau)|$$

- If $X(t)$ has a constant component, say, $X(t) = A + V(t)$, where $V(t)$ has zero mean, the autocorrelation function has a constant component.

$$E\{[A+V(t)][A+V(t+\tau)]\} = E[A^2 + AV(t) + AV(t+\tau) + V(t)V(t+\tau)]$$

$$= A^2 + AE[V(t)] + AE[V(t+\tau)] + E[V(t)V(t+\tau)] = A^2 + R_V(\tau)$$

Properties of Autocorrelation Functions

- If $X(t)$ has a periodic component, then the autocorrelation function has a periodic component. $X(t) = A\cos(\omega t + \Theta)$

where A and ω are constant and Θ a r.v. uniformly distributed over (0,2π); i.e. $f_\Theta(\theta) = (2\pi)^{-1}, 0 \leq \theta < 2\pi; = 0,$ elsewhere.

$$R_X(\tau) = E[A\cos(\omega t + \Theta)A\cos(\omega t + \omega\tau + \Theta)]$$

$$= E\left[\frac{A^2}{2}\cos(2\omega t + \omega\tau + 2\Theta) + \frac{A^2}{2}\cos(\omega\tau)\right]$$

$$= \frac{A^2}{2}\cos(\omega\tau) + \frac{A^2}{2}\int_0^{2\pi}\frac{1}{2\pi}\cos(2\omega t + \omega\tau + 2\theta)\,d\theta = \frac{A^2}{2}\cos(\omega\tau)$$

- If $X(t)$ is ergodic and zero-mean, and has no periodic components,

$$\lim_{|\tau|\to\infty} R_X(\tau) = 0$$

That is, time samples far apart tend to behave statistically independently.

Properties of Autocorrelation Function

- Autocorrelation functions cannot have arbitrary shape – they must correspond to some power spectrum which must be non-negative over the entire frequency range. More discussions later.

$$\mathcal{F}\,[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}\,d\tau = \text{power spectrum of } X(t)$$

$$\mathcal{F}\,[R_X(\tau)] = S_X(\omega) \geq 0 \quad \text{for all } \omega$$

Example: An ergodic random process has an autocorrelation function of the form

$$R_X(\tau) = \frac{4\tau^2 + 6}{\tau^2 + 1}$$

Find the mean-square value, mean value, and variance of the process.

$$R_X(\tau) = \frac{4\tau^2 + 6}{\tau^2 + 1} = 4 + \frac{2}{\tau^2 + 1} \qquad \overline{X^2} = R_X(0) = 6$$

$$\overline{X}^2 = 4 \;\Rightarrow\; \overline{X} = \pm 2 \qquad \sigma_X^2 = \overline{X^2} - (\overline{X})^2 = 6 - 4 = 2$$