ECE 3075A
Random Signals

Lecture 2
Probability Theory and Set Algebra

School of Electrical and Computer Engineering
Georgia Institute of Technology
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Probability Theory

- Characterization of random events
- Approaches:
  – Relative frequency approach: event counting
  – Axiomatic approach: set theory
- Elementary set theory
- Probability space:
  – Observation or sample space
  – Set algebra
  – Probability assignment
  – Fundamental axioms

Some Notions

- An **experiment** is an action that produces **outcome**; for example, tossing a coin with H(ead) and T(ail) as possible outcomes. The set of all possible outcomes form the **observation space** of the experiment.
- A **random experiment** is one in which the outcome is uncertain until the experiment is performed.
- A single performance in an experiment is called a **trial.**
- A trial results in an observed outcome, which is a **realization** of the trial.
- An **event** is a subset of the observation space and may be defined by a description of outcomes; we say an event has occurred if the description of the event is observed in the realization of a trial.

Approaches to the Theory of Probability

- Relative frequency approach
  – An event that occurs more frequently has higher probability and vice versa
  – Empirical reasoning; readily understandable
  – Difficult to generalize
- Axiomatic approach
  – Probability is a real number between 0 and 1 assigned to events in the observation space
  – The number called probability satisfies a set of postulates so as to form a structure for further generalization and applications

Relative Frequency Approach

- Use the relative frequency of occurrences of an event as probability of the event.

Example:

A coin is tossed 1000 times, in which the outcome of "head" occurs 547 times and "tail" 453 times. We postulate that the probability of "head" is 0.547 and "tail" 0.453.

A dye is thrown 500 times among which "1" is observed 90 times, "2" 88 times, "3" 101 times, "4" 81 times, "5" 75 times and "6" 65 times. What would be a reasonable description of the probability of various outcomes?

Classical Definition of Probability

- The probability $\Pr(A)$ of an event A is determined *a priori*, **not through experimentation**, given as

$$\Pr(A) = N_A / N$$

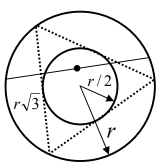
where N is the number of all possible outcomes (or cardinality of the observation space) and N_A is the number of outcomes that are favorable to the event A.

Example: In the game of die-throw, the number of all possible outcomes is 6 and the number of even outcomes is 3, resulting in $\Pr(even)$=0.5.

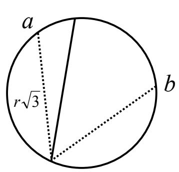
Question: Two dice are thrown. What is the probability that the sum of the two numbers equals 7?

Bertrand Paradox

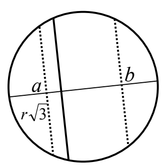
An equilateral triangle inscribed in a circle C of radius r has side length equal to $r\sqrt{3}$. Let us "randomly" draw a chord connecting two points on the circle. What is the probability that the length of the chord is greater than $r\sqrt{3}$?



Any chord with center landed in the inner circle will have length $> r\sqrt{3}$. The probability is thus the ratio of the areas of the inner and the outer circle, respectively, and is equal to ¼.



Any chord with one end located on the arc between a and b will have length $> r\sqrt{3}$. The probability is thus the ratio of the length of the arc ab and the perimeter of the circle, and is equal to 1/3.



Any parallel chord drawn between the two dotted lines will have length $> r\sqrt{3}$. The probability is thus the ratio of the length ab and the diameter of the circle, and is equal to 1/2.

Axiomatic Approach

- Built upon set theory
- A **Probability Space** consists of three components: the observation or sample space, S, set algebra, and the probability measure, Pr(), which assigns a probability to each and every event.
- The **Observation Space** is a space consisting of all the outcomes of a random experiment.
- **Events** are subsets of the observation space.
- Pr() must satisfy **three axioms**:

Non-negativity:	$\Pr(A) \geq 0$
Sure event:	$\Pr(S) = 1$
Exclusivity:	If $A \cap B = \phi$, then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$

Elementary Set Theory

- A set is a collection of objects known as elements.
- A space contains all the elements of interest, including the null element or empty set, ϕ , and the space, S itself.
- All sets formed by elements of the space are subsets of the space, S.

$a \in A \quad \Rightarrow \quad a$ belongs to (or is an element of) A;

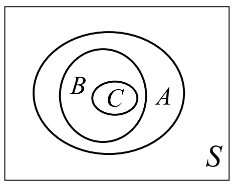
$A \subset B \quad \Rightarrow$ Set A is a subset of set B; or, B contains A

$A = B \quad iff \quad A \subset B$ and $B \subset A$.

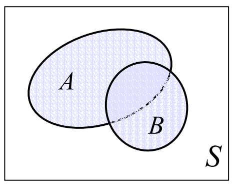
Set Operations

- Sum or Union: $A \cup B$ is a set that contains all the elements that are elements of A or of B **or** of both.
- Product or Intersection: $A \cap B$ is a set that contains all the elements that are common to both A **and** B.
- Complement: The complement of A, $\overline{A}$, is a set that contains all the elements of S that are not in A.
- Difference: The difference of two sets, $A - B$, is a set consisting of the elements of A that are not in B.

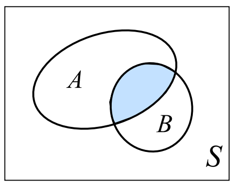
Venn Diagram



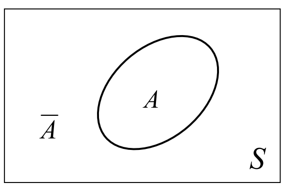
$C \subset B \subset A \subset S$



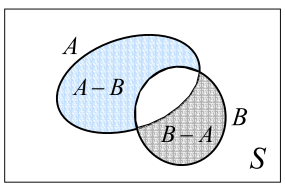
$A \cup B$



$A \cap B$



$\overline{A}$



$A - B$ and $B - A$