ECE 3075A
Random Signals

Lecture 15

Distributions of Functions of Several Random Variables

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Transformation of Multiple Random Variables

$Z=\varphi_1(X,Y)$ and $W=\varphi_2(X,Y)$ are two functions of r.v. X and Y.

Both φ_1 and φ_2 are continuous functions with corresponding inverse functions $X=\psi_1(Z,W)$ and $Y=\psi_2(Z,W)$, respectively.

Since all the events that map to $\{x_1<X(\xi)\le x_2, y_1<Y(\xi)\le y_2\}$ would also map to $\{z_1<Z(\xi)\le z_2, w_1<W(\xi)\le w_2\}$, we have

$$\Pr\{x_1<X\le x_2, y_1<Y\le y_2\}=\Pr\{z_1<Z\le z_2, w_1<W\le w_2\}.$$

or $$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{X,Y}(x,y)\,dydx=\int_{z_1}^{z_2}\int_{w_1}^{w_2} f_{Z,W}(z,w)\,dwdz$$

But $$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{X,Y}(x,y)\,dydx=\int_{z_1}^{z_2}\int_{w_1}^{w_2} f_{X,Y}(\psi_1(z,w),\psi_2(z,w))\,|J|\,dwdz$$

by way of change of variables with $$J=\begin{vmatrix}\frac{\partial x}{\partial z} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w}\end{vmatrix}=\begin{vmatrix}\frac{\partial \psi_1}{\partial z} & \frac{\partial \psi_1}{\partial w}\\ \frac{\partial \psi_2}{\partial z} & \frac{\partial \psi_2}{\partial w}\end{vmatrix}$$

J is the Jacobian that relates the incremental area $dzdw$ to $dxdy$.

Transformation of Multiple R.V. (cont'd)

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{X,Y}(x,y)\,dydx=\int_{z_1}^{z_2}\int_{w_1}^{w_2} f_{X,Y}(\psi_1(z,w),\psi_2(z,w))\,|J|\,dwdz$$

$$=\int_{z_1}^{z_2}\int_{w_1}^{w_2} f_{Z,W}(z,w)\,dwdz$$

Therefore, $f_{Z,W}(z,w)=|J|\,f_{X,Y}\big(\psi_1(z,w),\psi_2(z,w)\big)$

Example: $Z=XY,\quad W=X \;\Rightarrow\; X=W,\quad Y=Z/W$

$$J=\begin{vmatrix}\frac{\partial x}{\partial z} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w}\end{vmatrix}=\begin{vmatrix}0 & 1\\ w^{-1} & -\frac{z}{w^2}\end{vmatrix}=-w^{-1}$$

Thus, $f_{Z,W}(z,w)=\frac{1}{|w|}f_{X,Y}\left(w,\frac{z}{w}\right)$

The mariginals, $$f_Z(z)=\int_{-\infty}^{\infty}\frac{1}{|w|}f_{X,Y}\left(w,\frac{z}{w}\right)dw$$

$$f_W(w)=\int_{-\infty}^{\infty}\frac{1}{|w|}f_{X,Y}\left(w,\frac{z}{w}\right)dz$$

Transformation of Multiple R.V. - Example

Two random variables X and Y have a joint probability density function of the form

$$f_{X,Y}(x,y)=\begin{cases}1, & 0\le x,y\le 1\\ 0, & \text{elsewhere}\end{cases}$$

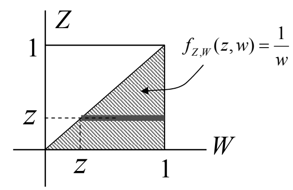
Find the pdf of $Z=XY$.

Use the result of the previous example:

$Z=XY,\quad W=X \;\Rightarrow\; X=W,\quad Y=Z/W$

$$f_{Z,W}(z,w)=\frac{1}{|w|}f_{X,Y}\left(w,\frac{z}{w}\right)=\frac{1}{w}\qquad 0\le w\le 1 \text{ and } 0\le z\le w$$

$$f_Z(z)=\int_z^1\frac{1}{w}dw=\ln w\Big|_{w=z}^{1}=-\ln(z)$$



Transformation of Multiple R.V. - Example

Let $Z = aX + bY$, $W = cX + dY$ Then, $X = (dZ - bW)/(ad - bc)$, $Y = (-cZ + aW)/(ad - bc)$

assume $ad - bc = A \neq 0$

The Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} d/A & -b/A \\ -c/A & a/A \end{vmatrix} = \frac{1}{A}$$

$$f_{Z,W}(z,w) = \frac{1}{|ad-bc|} f_{X,Y}\left(\frac{dz-bw}{ad-bc}, \frac{-cz+aw}{ad-bc}\right)$$

But, often it's not as complicated as it seems.

$$E[ZW] = E[acX^2 + bdY^2 + (ad+bc)XY] = ac\overline{X^2} + bd\overline{Y^2} + (ad+bc)\overline{XY}$$

$$E[Z^2] = E[a^2X^2 + b^2Y^2 + 2abXY] = a^2\overline{X^2} + b^2\overline{Y^2} + 2ab\overline{XY}$$

First and Second Order Moments

If we are only to find the first and second order moment of a function of several random variables, we may not need to find the joint density function. Consider $X = \sum_{i=1}^{N} a_i X_i$, , a weighted sum of several r.v.s.

$$\overline{X} = E[X] = E\left[\sum_{i=1}^{N} a_i X_i\right] = \sum_{i=1}^{N} a_i E[X_i] = \sum_{i=1}^{N} a_i \overline{X_i}, \quad X - \overline{X} = \sum_{i=1}^{N} a_i X_i - \sum_{i=1}^{N} a_i \overline{X_i} = \sum_{i=1}^{N} a_i (X_i - \overline{X_i})$$

$$\sigma_X^2 = E[(X-\overline{X})^2] = E\left[\sum_{i=1}^{N} a_i (X_i - \overline{X_i}) \sum_{j=1}^{N} a_j (X_j - \overline{X_j})\right]$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j E\left[(X_i - \overline{X_i})(X_j - \overline{X_j})\right] = \sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j C_{X_i X_j}$$

where $C_{X_i X_j} = E\left[(X_i - \overline{X_i})(X_j - \overline{X_j})\right]$ is the covariance between X_i and X_i

If these r.v.s are uncorrelated, $C_{X_i X_j} = \begin{cases} 0, & i \neq j \\ \sigma_{X_i}^2, & i = j \end{cases}$ Then, $\sigma_X^2 = \sum_{i=1}^{N} a_i^2 \sigma_{X_i}^2$

The variance of a weighted sum of uncorrelated random variables equals the weighted sum of the variances of the random variables (squared weights).