**ECE 3075A**
***Random Signals***

**Lecture 10**

# Expectations, and Moments

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## Mean Values and Moments

- Mean or expected value of a random variable

$$\bar{X} = E[X] = \int_{-\infty}^{\infty} x f_X(x)dx$$

- Mean or expected value of a function of a random variable $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)dx$

- $N^{th}$ moments & central moments of a r.v.

$$\overline{X^n} = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)dx \qquad \overline{(X-\bar{X})^n} = E[(X-\bar{X})^n] = \int_{-\infty}^{\infty} (x-\bar{X})^n f_X(x)dx$$

When $n = 2$,

$\overline{X^2} = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx$ is called the mean-square value

and $\sigma^2 = E[(X-\bar{X})^2] = \int_{-\infty}^{\infty} (x-\bar{X})^2 f_X(x)dx$ is the variance.

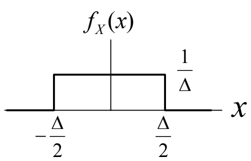
$$\sigma^2 = E[(X-\bar{X})^2] = E[X^2 - 2X\bar{X} + \bar{X}^2] = E[X^2] - 2\bar{X}^2 + \bar{X}^2 = E[X^2] - \bar{X}^2$$



## Moments of Uniform R.V.

$$f_X(x) = \frac{1}{\Delta}[u(x+\frac{\Delta}{2}) - u(x-\frac{\Delta}{2})]$$

$$u(n) = \text{"step function"} = \begin{cases} 1, & n \ge 0 \\ 0, & n < 0 \end{cases}$$




$$\bar{X} = \int_{-\Delta/2}^{\Delta/2} x f_X(x)\,dx = \int_{-\Delta/2}^{\Delta/2} \frac{x}{\Delta}dx = \frac{x^2}{2\Delta}\Big|_{-\Delta/2}^{\Delta/2} = \frac{\Delta}{8} - \frac{\Delta}{8} = 0$$

$$\overline{X^2} = \int_{-\Delta/2}^{\Delta/2} x^2 f_X(x)\,dx = \int_{-\Delta/2}^{\Delta/2} \frac{x^2}{\Delta}dx = \frac{x^3}{3\Delta}\Big|_{-\Delta/2}^{\Delta/2} = \frac{\Delta^2}{24} + \frac{\Delta^2}{24} = \frac{\Delta^2}{12}$$

$$\sigma^2 = \overline{X^2} - (\bar{X})^2 = \overline{X^2} = \frac{\Delta^2}{12} \qquad \overline{X^3} = \int_{-\Delta/2}^{\Delta/2} \frac{x^3}{\Delta}dx = \frac{x^4}{4\Delta}\Big|_{-\Delta/2}^{\Delta/2} = 0$$

The $n^{th}$ moment is zero for odd $n$.



## Example – Exercise 2-4.2

A random variable $X$ has a pdf of the form

$$f_X(x) = 0.25[u(x) - u(x-4)]$$

For the r.v. $Y$=$X^2$, find

a) The mean value

$$\bar{Y} = \int_0^4 x^2 \bullet 0.25\,dx = x^3/12\Big|_{x=0}^4 = 16/3$$

b) The mean-square value

$$\overline{Y^2} = \int_0^4 x^4 \bullet 0.25\,dx = x^5/20\Big|_{x=0}^4 = 256/5$$

c) The variance.

$$\sigma_Y^2 = \overline{(Y-\bar{Y})^2} = E[Y^2] - \bar{Y}^2 = (256/5) - (16/3)^2$$

$$= 256\left(\frac{1}{5} - \frac{1}{9}\right) = \frac{1024}{45}$$

## Entropy – Average Information

- Let $\{p_i\}_{i=1}^{L}$ be the (a priori) probability associated with a source which puts out symbols $\{X = x_i\}_{i=1}^{L}$ ;
- Define information in symbol $x_i$ as $-\log_2 p_i$ ;
- The average information is called entropy and is defined as

$$H = E_X[-\log_2 p_i] = -\sum_{i=1}^{L} p_i \log_2 p_i$$

For example, if $L = 2$, and $x_1 = 0$, and $x_2 = 1$, then it is a binary source with symbols 0 and 1. And if $p_1 = p_2 = 0.5$, then every symbol the source puts out carries one **bit** of information.
if $p_1 \neq p_2$, say, $p_1 = 0.8$ and $p_2 = 0.2$, then the average information in each symbol is less than one bit.



## Conditional Probability Distribution

We define the conditional probability the same as before.

$$F_X(x \mid M) = \Pr(X \le x \mid M) = \frac{\Pr(X \le x, M)}{\Pr(M)}, \quad \Pr(M) > 0$$

If we use the event mapping concept, $\Pr(X \le x, M)$ is the probability of all the outcomes which realize both events $X(\xi) \le x$ and $\xi \in M$.

If $M = \{X \le m\}$, $F_X(x \mid M) = \Pr(X \le x \mid X \le m) = \dfrac{\Pr(X \le x, X \le m)}{\Pr(X \le m)}$

If $x \le m$, $F_X(x \mid M) = \dfrac{\Pr(X \le x)}{\Pr(X \le m)} = \dfrac{F_X(x)}{F_X(m)}$ If $x \ge m$, $F_X(x \mid M) = \dfrac{\Pr(X \le m)}{\Pr(X \le m)} = 1$

Conditional probability density function has all the properties of a usual pdf. $f_X(x \mid M) = \dfrac{dF_X(x \mid M)}{dx}$



## Conditional Expected Value

- The expected value of $X$, given event $M$, is

$$E[X \mid M] = \int_{-\infty}^{\infty} x f_X(x \mid M)\, dx$$

if $M = \{X \le m\}$, $f_X(x \mid X \le m) = \begin{cases} \dfrac{f_X(x)}{F_X(m)}, & x < m \\ 0, & x \ge m \end{cases}.$

Thus, $E[X \mid X \le m] = \dfrac{\int_{-\infty}^{m} x f_X(x)\, dx}{\int_{-\infty}^{m} f_X(x)\, dx}$

which is the expected value of $X$ when $X$ is constrained to the set/event $\{X \le m\}$.



## Useful Inequalities

- Chebychev Inequality

$$\Pr\{|X - \overline{X}| \ge \varepsilon\} \le \sigma_X^2 / \varepsilon^2$$

$$\Pr\{|X - \overline{X}| \ge \varepsilon\} = \int_{-\infty}^{\overline{X}-\varepsilon} f_X(x)\, dx + \int_{\overline{X}-\varepsilon}^{\infty} f_X(x)\, dx = \int_{x \in B} f_X(x) dx$$

where $B = \left\{R - \{|x - \overline{X}| < \varepsilon\}\right\}$

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x - \overline{X})^2 f_X(x)\, dx \ge \int_{x \in B} (x - \overline{X})^2 f_X(x)\, dx \ge \varepsilon^2 \int_{x \in B} f_X(x)\, dx$$
$$= \varepsilon^2 \Pr\{|X - \overline{X}| \ge \varepsilon\}$$

- Markov Inequality: for non-negative r.v. $X$,

$$\Pr\{X \ge a\} \le E[X]/a, \quad a > 0$$