On stability number of some Markovian random graphs

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Background

In a graph G with vertex set V and edge set E, a subset $A \subseteq V$ is independent if no pair of two elements in A is connected i.e.

$$\forall a_1, a_2 \in A, \quad (a_1, a_2) \notin E$$

- The stability number denoted by $\alpha(G)$ of G is the maximum size of independent set.
- For a Erdös-Rényi random graph model $G(n,p)$ in which edge variable are i.i.d. Bernoulli(p). Its stability number [5, 6, 10] is well studied in the following two cases.

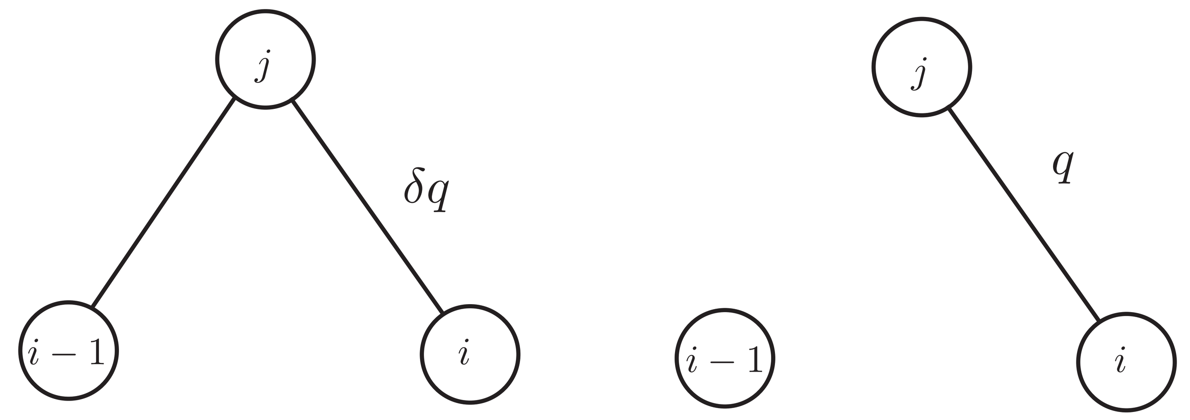
$$\alpha(G(n,p)) \approx 2\log_{\frac{1}{1-p}} n, \quad p \text{ is constant}$$

$$\alpha(G(n,p)) \approx \frac{2n}{d}\log d, \quad d := np \text{ is constant}$$

- We are interested in the graph with non-i.i.d. edge variables as dependencies are common in real-life models as social networks [3]. The hardness of analyzing such model is not surprisingly rooted into the dependency structure of edges.

Markov Graph Model

A Markov $G_\delta^M(n,p)$ is a model in which the probability of realizing edge (i,j) with $i<j$ depends on but only on the presence of its previous edge $(i-1,j)$. So edges are dynamically generated in a Markov process. Here $\delta \in (0,1)$ is a decay factor and p is the initial state parameter.



The left graph edge $(i-1,j)$ is present, the probability of generating (i,j) is δq where $q := P\{X_i^{(j)} = 1\}$ and $X_i^{(j)}$ is the variable of edge $(i-1,j)$.

Stability Number

Asymptotic stability number

Theorem. *(ASN) For $\lambda > 1$, with high probability,*

$$\alpha(G_\delta^M(n,p)) \leq \left(1 + \frac{2}{3e} - e^{-h}\right)\cdot n$$

where $h = \frac{\delta}{\lambda(1-\delta)}$.

Greedily Independent Set

Greedy stability number

Theorem. *Let $\alpha^G(G_\delta^M(n,p))$ to be size of the maximal independent set returned by greedy algorithm, then with high probability*

$$\alpha^G(G_\delta^M(n,p)) = \Omega\left(n^{\frac{1}{w+1}}\right)$$

where $w = \lceil \frac{1}{1-\delta} \rceil$ and Ω means "asymptotically larger".

It's clear that $\alpha^G(G_\delta^M(n,p)) \leq \alpha(G_\delta^M(n,p))$ and therefore, this result provides a lower bound on the stability number of $G(p,t)$.

Vertex Degree

$\deg v_i$ and $\deg G_\delta^M(n,p)$ are the degrees of vertex v_i and the average vertex degree of the graph respectively.

Asymptotic convergence of degree

Theorem. *For $\epsilon > 0$, with high probability,*

$$\left|\frac{\deg v_i}{n} - \frac{1}{(1-\delta)i}\right| \leq \frac{\epsilon}{(1-\delta)i}$$

$$\left|\frac{\deg G_\delta^M(n,p)}{\log n} - \frac{2}{1-\delta}\right| \leq \frac{\epsilon}{1-\delta}$$

Proof Sketch of ASN

The main technique is applying Chebyshev inequality on $H_{k,n}$ the number of independent sets of size k in $G_\delta^M(n,p)$.

$$P\{\alpha(G_\delta^M(n,p)) > k\} \leq P\{H_{k,n} > 0\} \leq \mathbb{E}[H_{k,n}]$$

Our target is to find $\max\{k : H_{k,n} = 0\}$ for each n. The idea is to evaluate the probability of a subset of k vertices to be independent since

$$\mathbb{E}[H_{k,n}] = \sum_{|A|=k:A\subseteq V} P\{A \text{ is independent}\}$$

Independent Set

Let A be a subset of k vertices in $G_\delta^M(n,p)$.

Proposition. *We say v_{n+1} is disconnected from A if there is no edge present between any vertex in A and v_{n+1}.*

$$(1-p)\prod_{j=1}^{k-1}(1-x_j) \leq z_{t+1}(A) \leq \prod_{j=t+1-k}^{t}(1-\delta x_j)$$

where $z_{t+1}(A) := P\{v_{t+1} \text{ is disconnected from } A\}$

This is the first ingredient of our proof and it leads to the bound on greedy stability number. Furthermore, this proposition implies an upper bound on the probability A to be independent.

Proposition. *Let $h = \frac{\delta}{\lambda(1-\delta)}$ and $a := \phi_\lambda + t$,*

$$P\{A \text{ is independent}\} \leq \left(e^{m-1}\cdot\left(1 - \frac{m-1}{a}\right)^a\right)^h$$

Proof Sketch of Degree

Follow from graph construction, we split edges connecting v_i into two parts.

$$\deg v_i = \sum_{j=1}^{i-1} X_j^{(i)} + \sum_{j=i+1}^{n} X_i^{(j)} \qquad (1)$$

The left part is the sum of first $i-1$ states in Markov chain while the right-hand side is the sum of i-th states from different Markov processes and they are pair-wisely independent since we do not assume dependency between chains. Let x_i denote the success probability of i-th state in Markov chain.

Proposition. *Fix $\lambda > 1$,*

$$\forall i \geq 2, \quad \frac{1}{\phi_\lambda + \lambda i} \leq (1-\delta)x_i < \frac{1}{i}$$

where $\phi_\lambda = \max\{\frac{\lambda}{\lambda-1}, \frac{1}{(1-\delta)p}\}$.

From law of big-number, the right hand side is approximately between $\frac{n}{\lambda i}$ and $\frac{n}{i}$. The left hand side (1) is however more complicated due to the variables therein are dependent. The idea is to use Markov inequality to get concentration result and its second moment can be bounded above.

Proposition.

$$\mathbb{E}[S_{i-1}^2] \leq \frac{2\delta}{1-\delta}\mathbb{E}[S_{i-1}] + \left(\frac{\log(i-1)}{1-\delta}\right)^2$$

where S_{i-1} denotes the sum of first $i-1$ states in our model Markov procedure.

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