Hyperbolic Relaxation of Locally PSD Matrices

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Key objects and goals

We study $\mathcal{S}^{n,k}$: the cone of symmetric matrices where all $k \times k$ principal submatrices are PSD.

What we want to understand about $\mathcal{S}^{n,k}$:

- The eigenvalues of matrices in $\mathcal{S}^{n,k}$, and whether the eigenvalue set has a convex relaxation.
- The quality and tightness of this convex relaxation, if it exists

Introduction

Positive semidefinite (PSD) matrices are of fundamental interest in a wide variety of fields, ranging from optimization to physics. Formally, a symmetric matrix M is PSD if and only if

$$u^\top M u \geq 0 \quad \text{for all } u \in \mathbb{R}^n.$$

We are interested in matrices that are locally PSD, where all of the $k \times k$ principal submatrices are PSD. We denote this set of matrices by $\mathcal{S}^{n,k}$. These are nested closed convex cones, with $\mathcal{S}^{n,1}$ being the matrices with nonnegative diagonal entries and $\mathcal{S}^{n,n}$ the usual PSD cone. An equivalent formulation of $\mathcal{S}^{n,k}$ is given by

$$\mathcal{S}^{n,k} = \{M : u^\top M u \geq 0 \text{ for all } k\text{-sparse } u \in \mathbb{R}^n\}.$$

This generalizes the second-order cone (SOC) relaxation of the PSD cone, which is the special case when $k = 2$.

We are interested in the unordered eigenvalue set of matrices in $\mathcal{S}^{n,k}$, which is invariant under permuting entries. We denote it by

$$\Lambda^{n,k} = \{\lambda(M) : M \in \mathcal{S}^{n,k}\}.$$

Background material

Given positive integers k, n, a homogeneous degree k polynomial p in n variables is said to be hyperbolic (in the all-one direction) if for all $x \in \mathbb{R}$, $p(t\mathbf{1} - x)$ is real-rooted as a univariate polynomial in t. All real stable polynomials are hyperbolic.

Given a hyperbolic polynomial p, its hyperbolicity cone, denoted by $H(p)$, is the set of x where all roots of $p(t\mathbf{1} - x)$ are nonnegative. It is known that $H(p)$ is always a closed convex cone.

Given integers $1 \leq k \leq n$, the degree k elementary symmetric polynomial in n variables, denoted by e_n^k, is defined as

$$e_n^k(x) = \sum_{|S|=k} \prod_{i \in S} x_i.$$

All elementary symmetric polynomials are hyperbolic. For fixed n their hyperbolicity cones are permutation invariant, and satisfy $H(e_n^1) \supseteq H(e_n^2) \supseteq \ldots \supseteq H(e_n^n)$, with $H(e_n^1)$ being the halfspace $\{x : \sum_i x_i \geq 0\}$ and $H(e_n^n)$ the nonnegative orthant $\{x : x_i \geq 0, \forall i\}$.

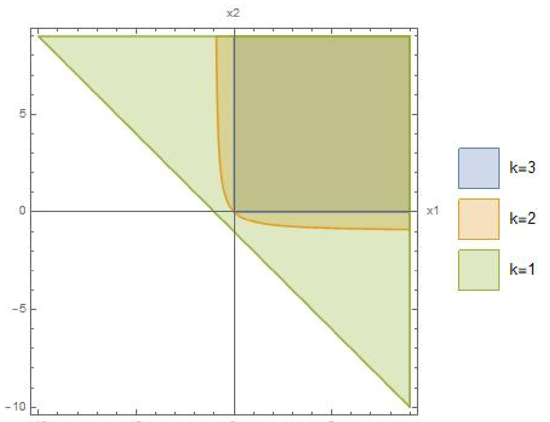

Figure 1:Intersections of $H(e_3^k)$ with $\{x_3 = 1\}$ plane

Linear optimization on affine sections of hyperbolicity cones is known as hyperbolic programming. It is polynomially solvable, and generalizes semidefinite programming (SDP).

Theorems and results

Theorem 1. Fix integers $1 \leq k \leq n$. Then $\Lambda^{n,k} \subseteq H(e_n^k)$. Equality holds if and only if $k = 1, n-1$ or n.

Thus the hyperbolicity cone of elementary symmetric polynomials is a convex relaxation of eigenvalues of matrices in $\mathcal{S}^{n,k}$. This allows us to solve the minimum eigenvalue problem under a variety of normalization constraints.

Theorem 2. Fix integers $2 \leq k \leq n$. Let $F : \mathbf{Sym}_n \to \mathbb{R}$ be a unitarily invariant matrix norm or the trace function. Let $G(n,k) = kI - \mathbf{1}\mathbf{1}^\top$. Then $\lambda_{\min}(M) \geq \lambda_{\min}(G(n,k))$ for all $M \in \mathcal{S}^{n,k}$ such that $F(M) = F(G(n,k))$.

This theorem shows that $G(n,k)$ achieves the most negative eigenvalue among all matrices in $\mathcal{S}^{n,k}$, subject to trace, Frobenius norm, Schatten p-norm or Ky Fan norm normalization constraint.

When $2 \leq k \leq n-2$ and $\Lambda^{n,k} \subsetneq H(e_n^k)$, we prove a structure theorem on nonsingular matrices in $\mathcal{S}^{n,k}$ whose eigenvalues are on the boundary of hyperbolic relaxation.

Theorem 3. Fix integers $2 \leq k \leq n-2$. Let $M \in \mathcal{S}^{n,k}$ be nonsingular with $\lambda(M) \in \partial H(e_n^k)$.

- If $k = 2$ and $n \geq 5$, then M is diagonally congruent to a ± 1 symmetric matrix with diagonal entries all one. Furthermore, M has at most $n-3$ negative eigenvalues.
- If $3 \leq k \leq n-2$ or $(n,k) = (4,2)$, then M is diagonally congruent to $G(n,k) = kI - \mathbf{1}\mathbf{1}^\top$. In particular, M has only one negative eigenvalue.

Note that for all $1 \leq k \leq n$, $\partial H(e_n^k)$ contains elements with $n-k$ negative entries.

Conclusion

Implication of our results

We obtained a convex relaxation to the eigenvalue set of locally PSD matrices. On this relaxation, general linear optimization can be done in polynomial time, and some optimization problems can be done directly and are provably exact.

We gave exact characterization on tightness of this relaxation based on n and k. When it is not tight, we proved that most of their boundaries do not intersect.

Current and future research

We conjecture that $\Lambda^{n,k}$ is not convex in general. Proof for $(n,k) = (4,2)$ case was completed by Kozhasov.

We also conjecture that all elements (containing no zero entries) on $\partial H(e_n^k)$ with correct signs indicated by Theorem 3 can be achieved as eigenvalues of some matrix in $\mathcal{S}^{n,k}$. We have proved this for $(n,k) = (4,2)$. On the other hand, eigenvalues of singular matrices in $\mathcal{S}^{n,k}$ are currently not well understood.

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Link to our paper

G. Blekherman, S. S. Dey, K. Shu, and S. Sun, "Hyperbolic relaxation of k-locally positive semidefinite matrices," submitted, 2020. https://arxiv.org/abs/2012.04031