ON PERFECT ROMAN DOMINATION IN GRAPHS

LEONARD M. PALETA[1] and FERDINAND P. JAMIL[2]

[1] *Mathematics Department, College of Science and Mathematics, University of Southern Mindanao, Kabacan, 9407 North Cotabato*

[2] *Department of Mathematics and Statistics, College of Science and Mathematics, Center for Graph Theory, Algebra and Analysis, Premier Research Institute of Science and Mathematics, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City*

ABSTRACT

In this paper, we continue the study of perfect Roman domination on the join, corona, complementary prism, edge corona and composition in graphs.

INTRODUCTION

In fourth century AD, for the defense of his cities, Emperor Constantine of Rome, decreed that any city without a legion stationed to secure it must neighbor another city having two stationed legions[2]. If the first were attacked, then the second could deploy a legion to protect it without becoming vulnerable itself. This new strategy is called ***defense-in-depth strategy***, which used only four field armies available for deployment to defend a total of eight regions. See Figure 1.

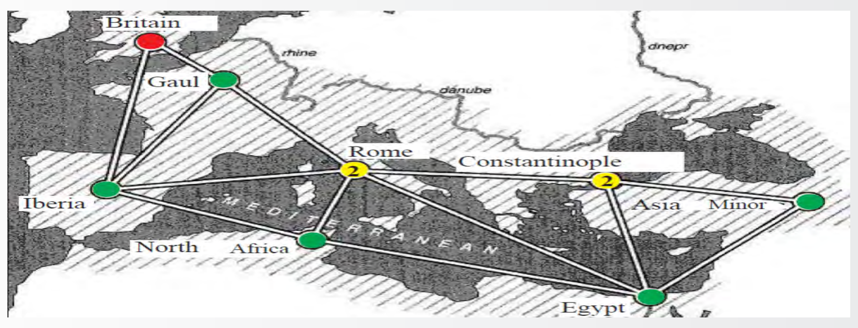

Figure 1: The Roman Empire, fourth century AD.

Constantine's strategy is now known in graph theory as ***Roman domination***. This mathematical concept was introduced by Cockayne, et al. [2] in 2004. This paper further explored the concept of perfect Roman domination in graphs introduced by Henning, et al[4] in 2018.

DEFINITIONS

Definition 1.[2]A *Roman dominating function* on G is a function $f : V(G) \to \{0,1,2\}$ such that for each $u \in V(G)$ for which $f(u) = 0$, there exists $v \in V(G)$ such that $f(v) = 2$ and $uv \in E(G)$. The *weight* of f is the value $\omega_G(f) = \sum_{v \in V(G)} f(v)$. The *Roman domination number* of G, denoted by $\gamma_R(G)$, is the minimum weight of a function f on G.

Customarily, we write $f = (V_0, V_1, V_2)$ for any function $f : V(G) \to \{0,1,2,\}$, where $V_k = \{v \in V(G) : f(v) = k\}$, $k \in \{0,1,2\}$. Hence, $f = (V_0, V_1, V_2)$ is a Roman dominating function of $G \iff$ for each $v \in V_0$, $|N_G(v) \cap V_2| \geq 1$.

Definition 2.[4] A *perfect Roman dominating function* (*PRD-function*) on G is a Roman domination function $f = (V_0, V_1, V_2)$ on G such that for each $u \in V_0$ there exists exactly one $v \in V_2$ for which $uv \in E(G)$. The *perfect Roman domination number* of G, $\gamma_R^P(G)$, is the minimum weight of a PRD-function on G. A PRD-function f with $\omega_G(f) = \gamma_R^P(G)$ is called γ_R^P*-function* of G.

Definition 3.[3] The *join* of two graphs G and H, denoted by $G+H$, is the graph with vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Definition 4.[3] The *corona* $G \circ H$ of G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i^{th} vertex of G to every vertex of the i^{th} copy of H.

Definition 5.[5] The *edge corona* $G \diamond H$ of G and H is the graph obtained by taking one copy of G and $|E(G)|$ copies of H and joining each of the end vertices u and v of each edge uv of G to every vertex of the copy H^{uv} of H.

Definition 6.[3] The *composition* $G[H]$ of two graphs G and H is the graph with vertex-set $V(G[H]) = V(G) \times V(H)$ and edge-set $E(G[H])$ satisfying the following conditions: $(x,u)(y,v) \in E(G[H])$ if and only if either $xy \in E(G)$ or $x = y$ and $uv \in E(H)$

Let P_2 and P_3 be the paths on 2 and 3 vertices, respectively.

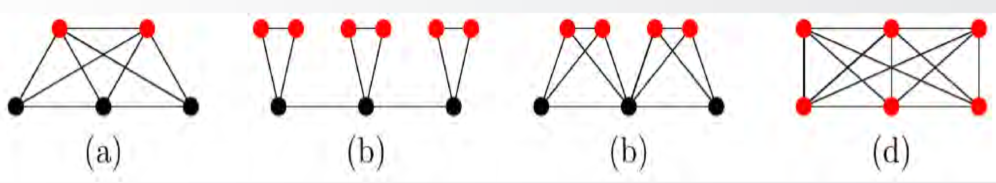

Figure 2:(a)$P_3 + P_2$,(b)$P_3 \circ P_2$, (c) $P_3 \diamond P_2$, (d) $P_3[P_2]$

Definition 7.[7]For a graph G, the *complementary prism*, denoted $G\overline{G}$, is formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$.

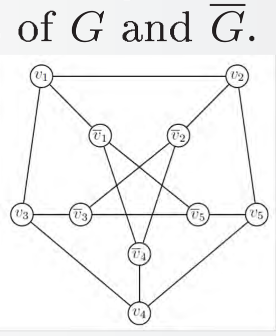

Figure 3: $C_5\overline{C_5}$ (Petersen graph).

MOTIVATION OF THE STUDY

Perfect Roman domination models many ***facility location problems*** [1], where $f(v)$ is viewed as cost function. Units with cost 2 may be able to serve neighboring locations, while units with costs 1 can serve only their own location.

In a ***communication network***, $f(v) = 2$ is assigned to locations where we install wireless hubs which are more expensive but can serve neighboring locations, while $f(v) = 1$ is assigned to locations where we install wired hubs which functions at low-range but are cheaper.

Another application is on the ***distribution of rare resources*** [8]: Given a graph, servers (resources) should be placed on nodes to service a pair of requests that can occur at nodes. Since resources are rare, the number of used resources should be minimized. Distributing resources is a challenging problem: Where should resources (e.g., schools, ambulances) be placed such that a desired property is fulfilled and the number of used resources is minimized?

One of the major applications of this study is on ***military strategies***[6]. History supports that General Douglas MacArthur, one of the greatest generals in the US Army, used the Roman domination strategy as *island-hopping strategy* in World War II in the Pacific theater.

MAIN RESULTS

For convenience, we adapt the symbol $PRD(G)$ to denote the set of all perfect Roman dominating functions on the graph G.

Theorem 1. Let G and H be any nontrivial connected graphs and $f = (V_0, V_1, V_2)$ on $V(G+H)$. Then $f \in PRD(G+H)$ if and only if one of the following holds:

(*i*) $V_2 \subseteq V(G)$ and one of the following holds: **(a)** $V_0 \subseteq V(G)$, $V(H) \subseteq V_1$ and $(V_0, V_1 \cap V(G), V_2) \in PRD(G)$; **(b)** $V_0 \cap V(H) \neq \varnothing$ and $V_2 = \{v\}$ for which $V_0 \cap V(G) \subseteq N_G(v)$.

(*ii*) $V_2 \subseteq V(H)$ and one of the following holds: **(a)** $V_0 \subseteq V(H)$, $V(G) \subseteq V_1$ and $(V_0, V_1 \cap V(H), V_2) \in PRD(H)$; **(b)** $V_0 \cap V(G) \neq \varnothing$ and $V_2 = \{v\}$ for which $V_0 \cap V(H) \subseteq N_H(v)$.

(*iii*) $A_1 = V_2 \cap V(G) \neq \varnothing$ and $A_2 = V_2 \cap V(H) \neq \varnothing$ and the following holds: **(a)**] If $V_0 \cap V(G) \neq \varnothing$, then $|A_2| = 1$ and $(V_0 \cap V(G)) \cap N_G(A_1) = \varnothing$; **(b)** If $V_0 \cap V(H) \neq \varnothing$, then $|A_1| = 1$ and $(V_0 \cap V(H)) \cap N_H(A_2) = \varnothing$.

Theorem 2. Let G and H be nontrivial graphs with G connected, and $f = (V_0, V_1, V_2)$ on $V(G \circ H)$. Then $f \in PRD(G \circ H)$ if and only if the following holds:

(*i*) For all $v \in V_0 \cap V(G)$ either **(a)** $V_2 \cap N_G(v) = \varnothing$ and $V_2 \cap V(H^v) = \{u\}$ with u satisfying $V_0 \cap V(H^v) \subseteq N_{H^v}(u)$; or **(b)** $|V_2 \cap N_G(v)| = 1$ and $V(H^v) \subseteq V_1$;

(*ii*) For all $v \in V_1 \cap V(G)$, the restriction $f|_{H^v}$ of f to H^v is a perfect Roman dominating function on H^v;

(*iii*) For all $v \in V_2 \cap V(G)$ for which $V_0 \cap V(H^v) \neq \varnothing$, $V_0 \cap N_{H^v}(V_2 \cap V(H^v)) = \varnothing$.

Theorem 3. For any graph G,

$$1 + \max\{\gamma_R^P(G), \gamma_R^P(\overline{G})\} \leq \gamma_R^P(G\overline{G}) \leq \rho,$$

where $\rho = \min\{\omega_G(f) + n - |V_2| : f = (V_0, V_1, V_2) \in PRD(G) \cup PRD(\overline{G}\}$.

For an $f \in PRD(G)$, we write for each $a, b \in \{0,1,2\}$, $E_{ab}(f;G) = \{uv \in E(G) : (f(u) = a \wedge f(v) = b) \vee (f(u) = b \wedge f(v) = a)\}$, where "$\wedge$" and "$\vee$" denote "and" and "or", respectively.

Theorem 4. Let G be a nontrivial connected graph and H any graph of order n. Then $\gamma_R^P(G \diamond H) \leq \alpha$, where

$$\alpha = \min_{g \in PRD(G)} (\omega_G(g) + A_{11} + n\,(A_{01} + A_{22} + A_{00}))$$

where $A_{11} = |E_{11}(g;G)|\gamma_R^P(H)$; $A_{01} = |E_{01}(g;G)|$; $A_{22} = |E_{22}(g;G)|$; $A_{00} = |E_{00}(g;G)|$ and this upper bound is sharp.

Example 1. Consider the graph $G \diamond P_3$ in Figure 4, where G is the caterpillar $ca(2,0,2)$ with the corresponding vertex labelling. The function g on $V(G)$ given by $g(x) = g(z) = 2$, $g(y) = 1$ and $g(x) = 0$ else is in $PRD(G)$. Since $E_{00} = E_{01} = E_{22} = E_{00} = \varnothing$, $\alpha \leq \omega_G(g) = 5$ so that $\gamma_R^P(G \diamond P_3) \leq 5$. Now, note that $\{x, z\}$ is the unique γ-set of $G \diamond P_3$.

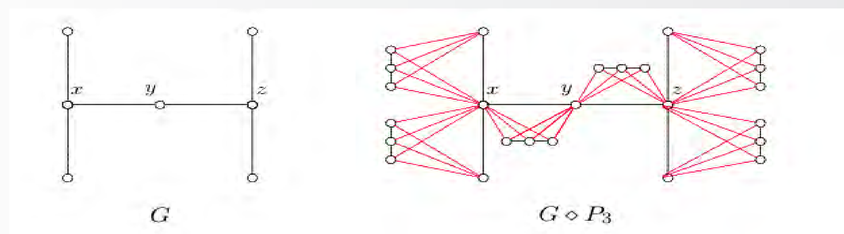

Figure 4: The edge corona $G \diamond P_3$ with $\gamma_R^P(G \diamond P_3) = 5$

Theorem 5. Let G and H be connected graphs, G noncomplete and H of order n with $\gamma(H) = 1$. Then

$$\gamma_R^P(G[H]) \leq \alpha,$$

where $\alpha = \min\{(n-1)(|V_1| + |V_2 \cap N_G(V_2)|) + \omega_G(f) : f = (V_0, V_1, V_2) \in PRD(G)\}$.

Theorem 6. Let G be a nontrivial connected graph and $p \geq 2$. Then

$$\gamma_R^P(G[K_p]) = \alpha,$$

where $\alpha = \min\{(n-1)(|V_1| + |V_2 \cap N_G(V_2)|) + \omega_G(f) : f = (V_0, V_1, V_2) \in PRD(G)\}$.

REFERENCES

[1] E.W.Chambers, B. Kinsley, N.Prince, D.B.West, *Extremal problems for Roman domination.* SIAM J Discrete Math 23:(2009) 1575-1586

[2] E. Cockayne, P. Dreyer Jr., S.M. Hedetniemi and S.T. Hedetniemi, *Roman domination in graphs*, Discrete Mathematics, Vol. 278 (2004), 11-22.

[3] T.W.Haynes, S.T.Hedetniemi, P.J.Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[4] M. A. Henning, W. F. Klostermeyer and G. MacGillivray. *Perfect Roman domination in trees*, Discrete Appl. Math. 236 (2018) 235–245.

[5] Y. Hou, and W. Shiu. *The spectrum of the edge corona of two graphs.* Electronic Journal of Linear Algebra, Vol. 20(2010) https://doi.org/10.13001/1081-3810.1395

[6] C.S. Revelle and K.E. Rosing. *Defendens imperium romanum: a classical problem in military strategy.* Amer. Math. Monthly, 107(7):(2000)585-594, .

[7] H.D.Russel, *Italian Domination in Complimentary Prisms*(2018).https://dc.etsu.edu/etd/3429

[8] K. Schulde. *Distributed data and resources models, tractability, and complexity.* ETH Zurich Research Collection. https://doi.org/10.3929/ethz-a-004484296

ACKNOWLEDGEMENT

Full paper is available at European Journal of Pure and Applied Mathematics, Vol. 13, No. 3, 2020, 529-548 ***[DOI: https://doi.org/10.29020/nybg.ejpam.v13i3.3763]****. This research was fully supported by the Commission on Higher Education(CHED) K-12 Transition Scholarship Program, Philippines*