Proximity in Concave Integer Quadratic Programming

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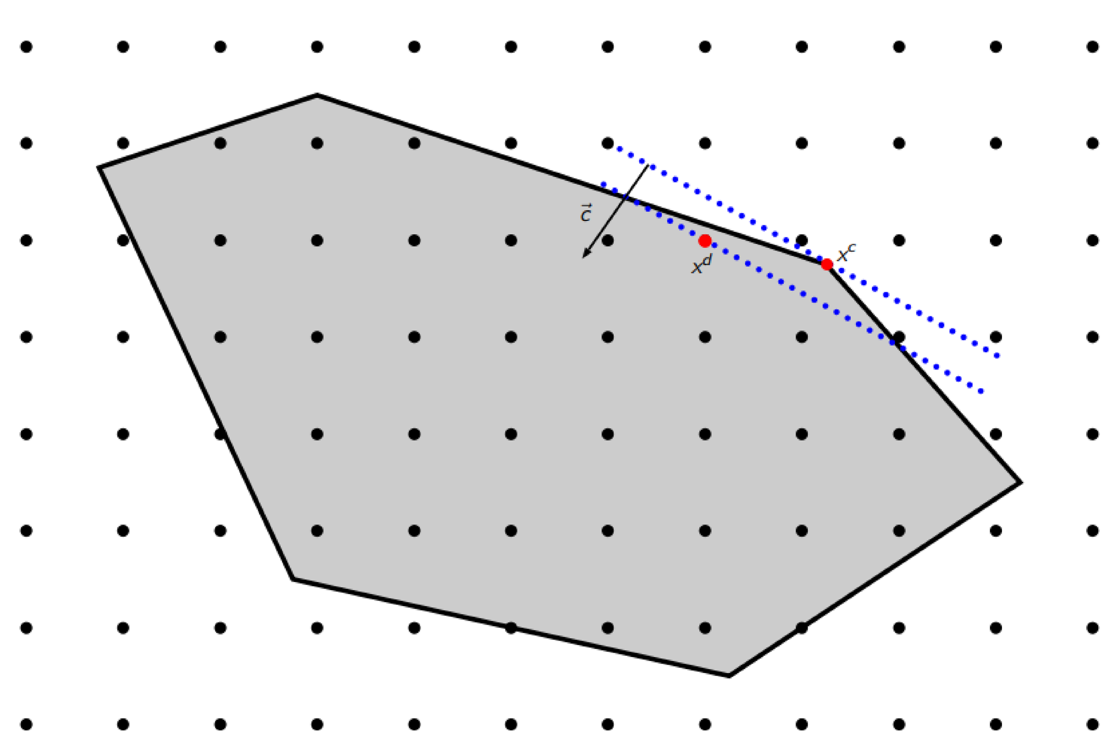
Proximity in Integer Optimization

- Let x^c be an optimal solution to $\min\{cx \mid Ax \leq b\}$.
- Let x^d be an optimal solution to $\min\{cx \mid Ax \leq b, x \in \mathbb{Z}^n\}$.
- An important question is to ask if there is an upper bound for $||x^c - x^d||_\infty$.



Theorem [Cook, Gerards, Schrijver and Tardos, 1986]

Denote by Δ, the largest absolute value of the subdeterminant of A.

- Let x^c be any optimal solution to (LP), then there is an optimal solution x^d to (IP) such that $||x^c - x^d||_\infty \leq n\Delta$.
- Let x^d be any optimal solution to (IP), then there is an optimal solution x^c to (LP) such that $||x^c - x^d||_\infty \leq n\Delta$.

- $n\Delta$ is still valid when minimize a convex separable quadratic function [Granot et al, 1990].
- $n\Delta$ is valid for a general convex separable function [Hochbaum et al,1990][Werman et al, 1991].
- New upper bound $p\Delta$ for mixed-integer linear programming [Paat, Weismantel, and Weltge, 2018].

Question: *Do proximity phenomena only occur in the presence of convexity?*

Concave Integer Quadratic Programming

$$\begin{aligned} \min\ & \sum_{i=1}^{k} -q_i x_i^2 + h^T x \\ \text{s.t. } & Ax \leq b \\ & x \in \mathbb{Z}^n. \end{aligned} \qquad \text{(IQP)}$$

$$\begin{aligned} \min\ & \sum_{i=1}^{k} -q_i x_i^2 + h^T x \\ \text{s.t. } & Ax \leq b \\ & x \in \mathbb{R}^n. \end{aligned} \qquad \text{(QP)}$$

- $q_i \geq 0$, A integral.
- Do proximity results happen for (IQP)?

A Counter Example

$$\begin{aligned} \min\ & f(x) = -\left(x - \frac{1}{4}\right)^2 \\ \text{s.t. } & -t \leq x \leq t + \frac{3}{4} \\ & x \in \mathbb{Z}. \end{aligned}$$

- t integer, n= 1, Δ= 1.
- $x^c = t + \frac{3}{4}, x^d = -t$.
- No proximity results if we consider optimal solutions.

ϵ-approximate solution

Definition ϵ-approximate solution

Let x^* be an optimal solution. For $\epsilon \in [0, 1]$, $x^\diamond$ is an ϵ-approximate solution if

$$\text{obj}(x^\diamond) - \text{obj}(x^*) \leq \epsilon(\text{obj}_{\max} - \text{obj}(x^*)).$$

- $\text{obj}(\cdot)$: objective function value.
- $\text{obj}_{\max}$: maximum value of obj(x) over the feasible region.
- Denition used in the literature from the 80s.
- Preserved under dilation and translation of obj.
- Insensitive to change of basis.

Main Result

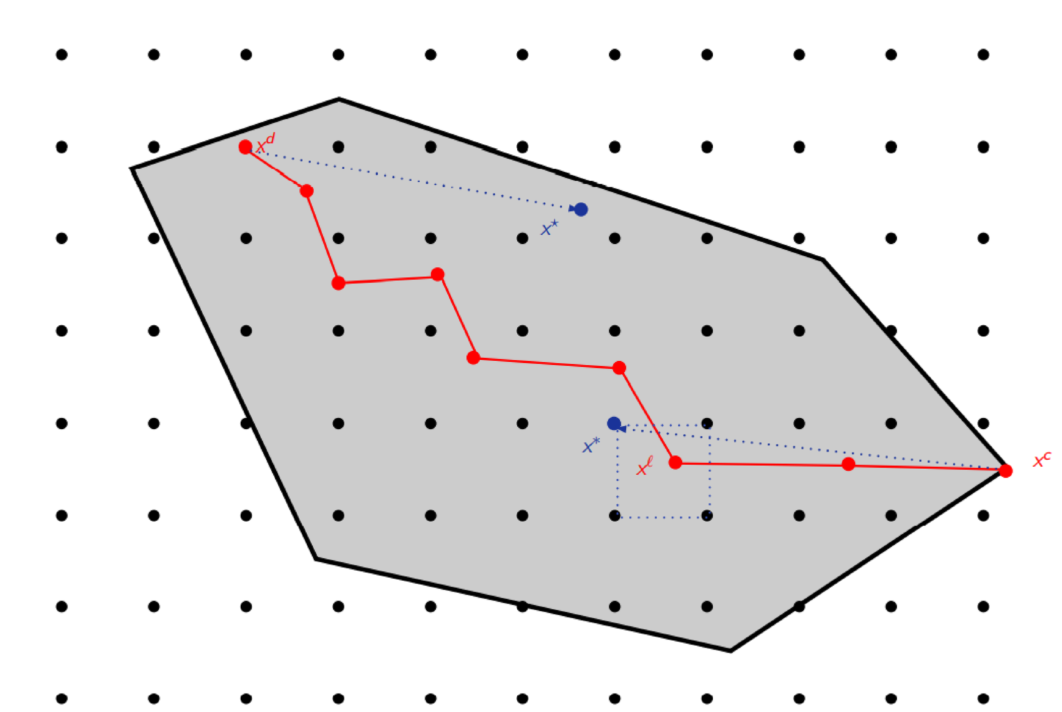
- We show that proximity phenomena still occur for concave integer quadratic programming.
- But only if we consider approximate solutions.

Theorem (Proximity in Concave Integer Quadratic Programming)

Consider a problem (IQP), and the corresponding continuous problem (QP). Suppose that both problems have an optimal solution. Then:

- Let x^c be any optimal solution to (QP). Then, $\forall\ \epsilon > 0$, there is an ϵ-approximate solution x^* to (IQP) such that
$$||x^* - x^c||_\infty \leq n\Delta\left(\frac{10\Delta}{\epsilon} + 1\right)^k.$$
- Let x^d be any optimal solution to (IQP). Then, $\forall\ \epsilon > 0$, there is an ϵ-approximate solution $x^\star$ to (QP) such that
$$||x^\star - x^d||_\infty \leq n\Delta\left(\frac{10\Delta}{\epsilon} + 1\right)^k.$$

- When $|x_i^* - x_i^d|$ is large for $i \in [k]$, x^* is a good approximation
- Based on x^c and x^d, we can construct a path with at most $k+1$ points inside the polyhedron.
- The length of the path can be bounded using n, Δ and ϵ.
- Either $x^* = x^d$ or x^* is near some point x^ℓ in the path.
- $x^\star$ can be found using x^c, x^d, x^*, and $||x^d - x^\star||_\infty = ||x^c - x^*||_\infty$.



Lower Bound for Proximity Result

- We use the following two quantities to describe the lower bound for proximity results.

$\delta_\epsilon^* := \min\{||x^c - x^*||_\infty | x^*\ \epsilon\text{-approx. to (IQP)},\ x^c \text{ opt. to (QP)}\}$,

$\delta_\epsilon^\star := \min\{||x^\star - x^d||_\infty | x^d \text{ opt. to (IQP)},\ x^\star\ \epsilon\text{-approx. to (QP)}\}$.

- We use $\bar{P}$ to obtain lower bounds for δ_ϵ^* and $\delta_\epsilon^\star$.
- $\delta_\epsilon^* \in \Omega(\frac{1}{\epsilon} + n\Delta)$.
- $\delta_\epsilon^\star \in \Omega(\frac{n\Delta}{\epsilon})$.
- $n\Delta$ bound for linear integer programming is asymptotically best possible according to $\bar{P}$.