Enumerating integer points in polytopes with bounded subdeterminants in polynomial time

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Statement & illustration of the algorithm

The max absolute value of any n-subdeterminant of A: $\mathbf{\Delta_n}$

Width of P in direction v: **w(v,P)** $:= max_{x\in P}\ vx - min_{x\in P}\ vx$

It is called **facet width** if v is a facet normal of P.

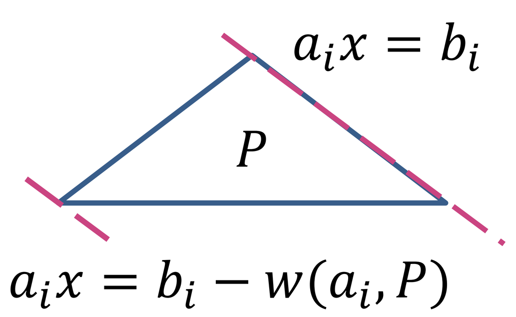

Figure 1(i)— Illustration of facet width.

INPUT: Polytope $P :=$ $\{x \in \mathbb{R}^n\colon Ax \le b\}$, where A, b integral and $\Delta_n \le \Delta$ (fixed). All n-subdeterminants of A are nonzero. The ith row (entry) of A (b) is denoted as a_i (b_i).

OUTPUT: Vertices of P's integer hull, P_I.

1. By [2], if $n > C(\Delta)$ for some $C\colon \mathbb{N} \rightarrowtail \mathbb{N}$. P can only be a simplex. When $n \le C(\Delta)$, we can use method in [1].
2. If $n > C(\Delta)$, check whether min facet width $< \Delta - 1$.
   - Y) Apply enumeration oracle on P.
   - N) Take $n+1$ simplices with small facet width at the corners. Then apply the enumeration oracle on each of them.

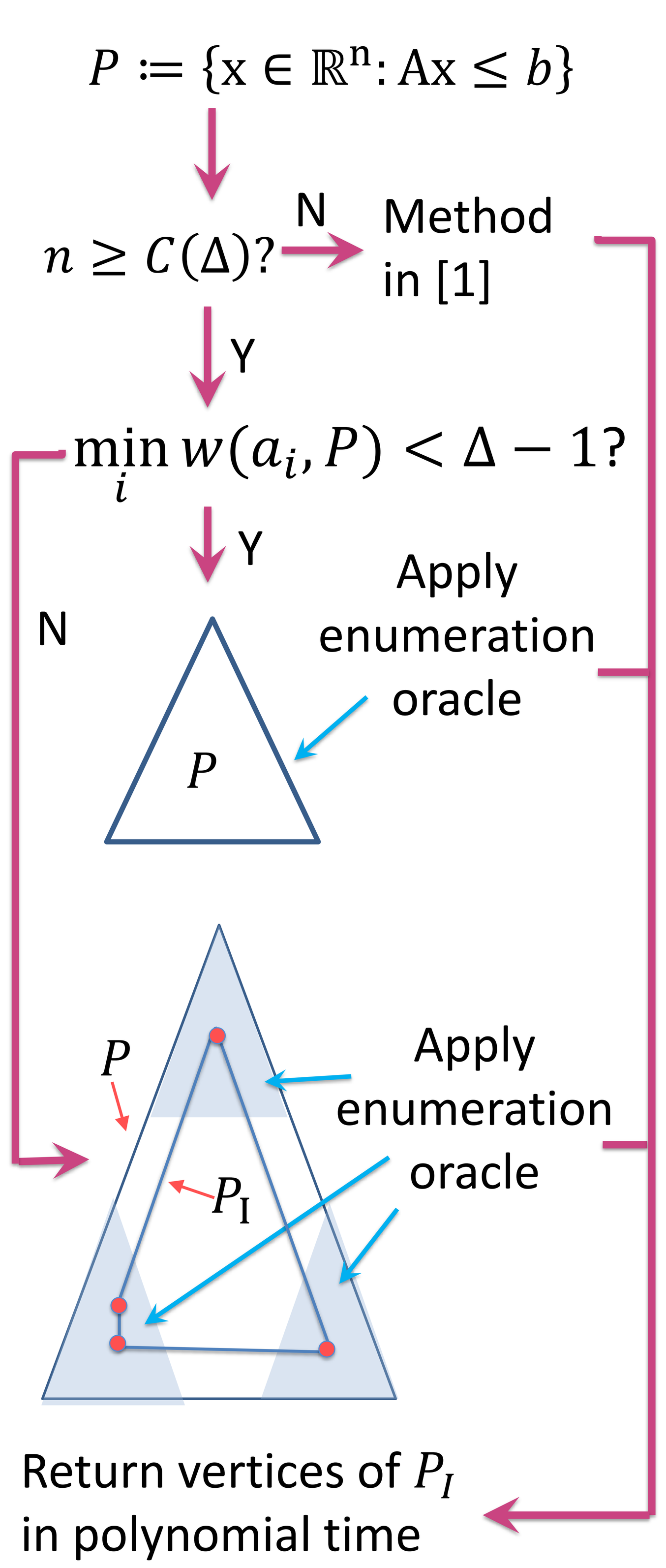

Figure 1(ii)— Illustration of the algorithm.

[1] Cook, William, et al. "On integer points in polyhedra." Combinatorica 12.1 (1992): 27-37.

[2] Artmann, Stephan, et al. "A note on non-degenerate integer programs with small sub-determinants." Operations Research Letters 44.5 (2016): 635-639.

Enumeration oracle for a 'small' simplex

INPUT: Simplex P with $w(a_i, P) \le W$ (fixed) for $1 \le i \le n$.

OUTPUT: All integer points in P.

Let the root node be P.

for $i\ =\ 0:\ (n\ -\ 1)$ **do**

1. For each nonempty node N at depth i, compute the width $W'\ = w(a_{i+1}, N)$.
2. Create the sets $N\ \cap\ \{x:$ $a_{i+1}x = b_{i+1} - w\}$ for $w \in \{0, \ldots, \lfloor W' \rfloor\}$ as children of N.

end for

Report all the nonempty leaf nodes that are integer points.

Intuition for polynomial complexity: As illustrated in Figure 2, N_2 and N_3 are translates of N_1, so their width in any direction v is shrunk by at least a factor of $\frac{W-1}{W}$ compared with N_1. Therefore, the number of their children is also shrunk by a constant factor compared with N_1. If we count all these translation effects in a recursive manner all the way through depth n of the tree, we can bound the number of leaves with a polynomial number.

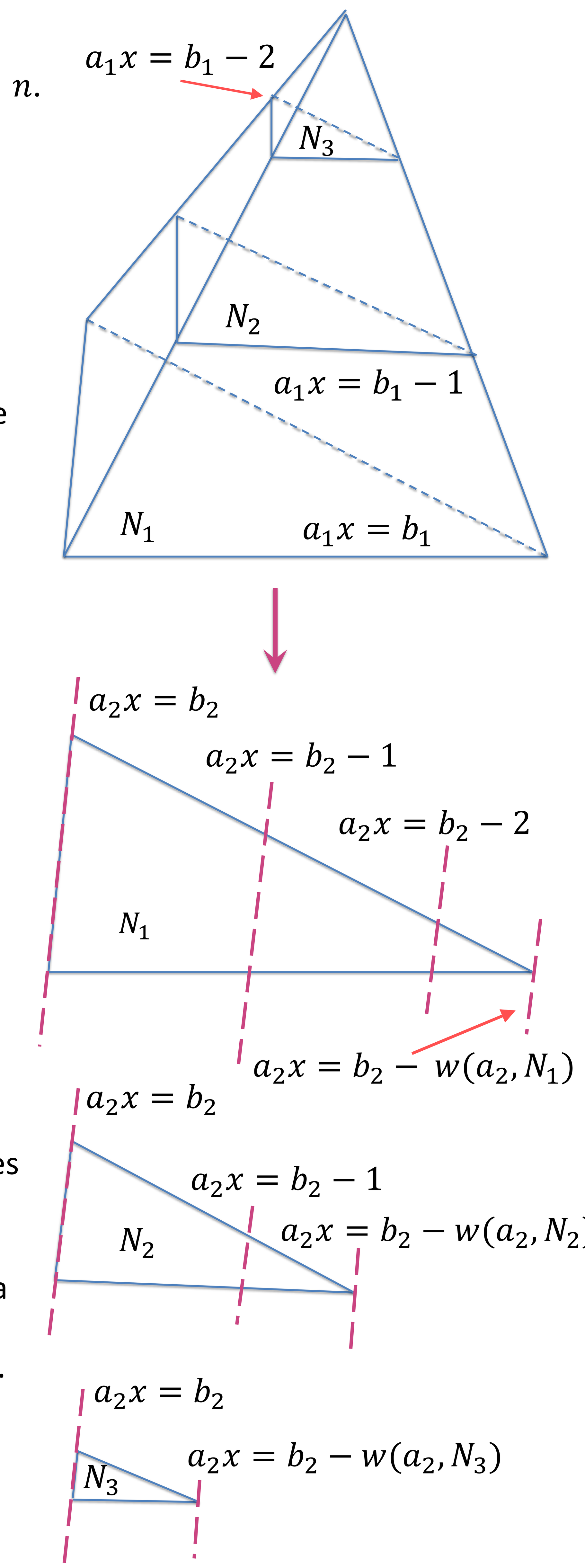

Figure 2— Illustration of part of the process of enumerating integer points in 'small' simplices. The children nodes of P are N_1, N_2 and N_3. There are three children nodes of N_1, two of N_2 and one of N_3.

Sketch of proof when Δ_n of a simplex is upper bounded by Δ

- Assume $\hat{A}$ is the inverse of the first n rows of A. Its ith column is denoted as $\hat{a}_i$.

Simplices with min facet width $< \mathbf{\Delta - 1}$:

- Sufficient to consider A in Hermite Normal Form with non-decreasing diagonals.

$\Delta_n \le \Delta$ ⇒

$$A = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ A_{i1} & \cdots & A_{i,i-1} & A_{ii} & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & \cdots & \vdots & \vdots & \cdots & A_{nn} \\ A_{n+1,1} & \cdots & A_{n+1,i-1} & A_{n+1,i} & \cdots & A_{n+1,n} \end{bmatrix}$$

At most $\lceil \log(\Delta) \rceil + 1$ rows

By [1], each abs(entry) is bounded by a constant.

$$\hat{A} = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hat{A}_{i1} & \cdots & \hat{A}_{i,i-1} & \hat{A}_{ii} & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ \hat{A}_{n1} & \cdots & \vdots & \vdots & \cdots & \hat{A}_{nn} \end{bmatrix}$$

At most $\lceil \log(\Delta) \rceil$ rows

Each abs(entry) is bounded by a constant.

$$\frac{w(a_i,P)}{w(a_j,P)} = \frac{a_{n+1}\hat{a}_i}{a_{n+1}\hat{a}_j} \le f(\Delta)$$

for $1 \le i, j \le n$ and some f: $\mathbb{N} \mapsto \mathbb{N}$.

min facet width ≤ Δ ⇒ max facet width ≤ $\Delta f(\Delta)$.

Simplices with min facet width $\ge \mathbf{\Delta - 1}$:

- Let C be the simplicial cone defined by the first n inequalities of $Ax \le b$. For each $v \in C$, there exists $\mu \in \mathbb{R}^n_{\ge 0}$, such that $v := p - \hat{A}\mu$, where p is the vertex of C. If $v \in C \cap \mathbb{Z}^n$, then $\mu \in \mathbb{Z}^n_{\ge 0}$.
- It can be proved that $S' := conv\{p, p - (\Delta-1)\hat{a}_1, \ldots, p - (\Delta-1)\hat{a}_n\}$ contains all the vertices of C's integer hull, C_I. Also, S' has first n facet width $= \Delta - 1$.
- For a 'large' simplex with $\Delta_n \le \Delta$, take $n+1$ such 'small' simplices at its $n+1$ corners, which include all P_I's vertices, and apply the enumeration oracle on each of them..

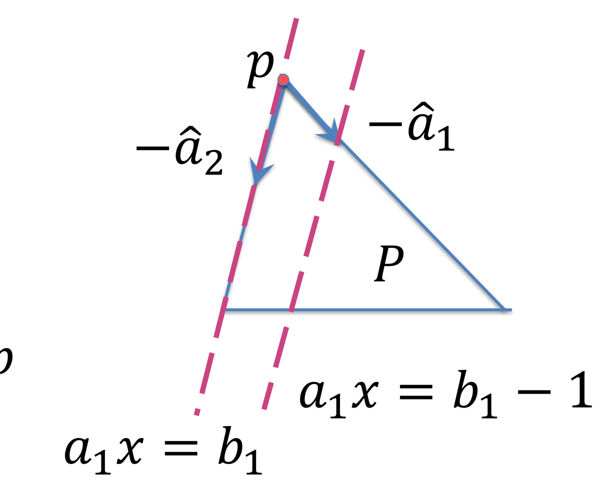

Figure 3(i) — Geometric meaning of $\hat{a}_i$.

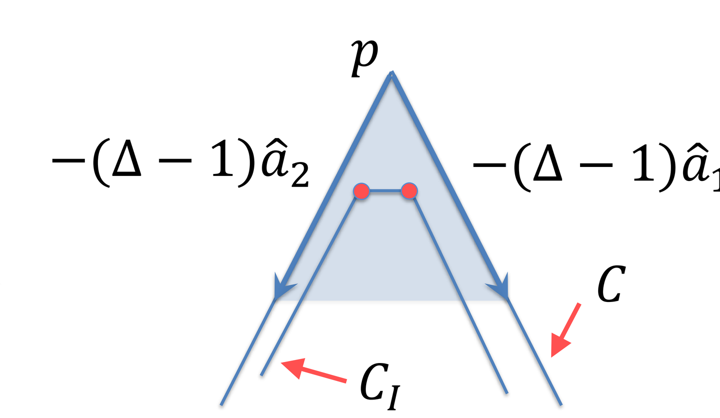

Figure 3(ii) — Truncated simplex.