The Price of Anarchy in Series-Parallel Network Congestion Games

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Network Congestion Games

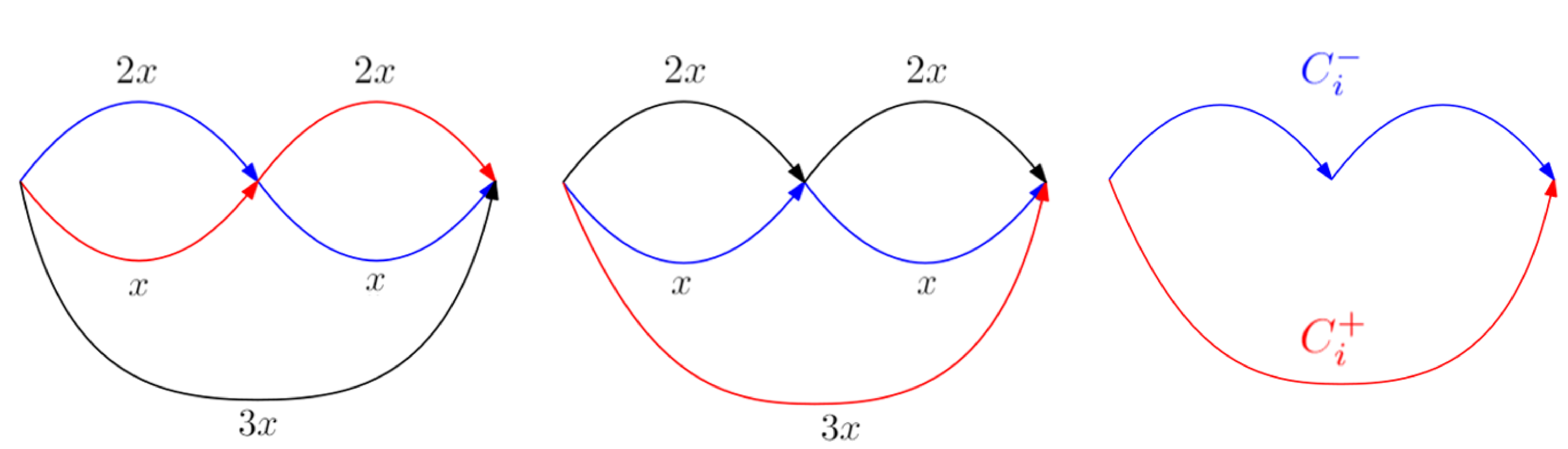
- N players
- An $(s,t)-$network $G=(V,E)$.
- $\forall$ player i, strategy set $X^i = \mathcal{P}$, the set of all $(s,t)-$paths.
- Set of states of the game $X = X^1 \times \cdots \times X^N$
- $\forall\ e \in E$ a nondecreasing delay function $d_e(x) = ax+b$, $a,b \geq 0$.

- Each state $(p^1,\dots,p^N) \in X$ induces an (s,t)-flow of value N in G.
- The cost of a flow g is $\text{cost}(g) = \sum_{e\in E} g_e d_e(g_e)$.
- The cost of a path p in G w.r.t. g is $\text{cost}_g(p) = \sum_{e\in p} d_e(g_e)$.
- The augmented cost of a path p in G w.r.t. g is $\text{cost}^+_g(p) = \sum_{e\in p} d_e(g_e+1)$.

- A pure Nash equilibrium (PNE) is a state $(p^1,\dots,p^i,\dots,p^N)$ inducing flow f such that, for each $i\in[N]$ we have
  $$\text{cost}_f(p^i) \leq \text{cost}_g(\tilde p^i) \qquad \forall (p^1,\dots,\tilde p^i,\dots,p^N) \in X \text{ inducing flow } g.$$
- A social optimum (SO) is a state inducing a flow o of minimum cost.
- The price of anarchy (PoA) is the ratio of cost of the most expensive PNE and cost of the SO.

Series Parallel Networks

An (s,t)-network is series-parallel if it consists of either a single edge (s,t) or of two series-parallel networks composed either in series or in parallel.



PNE flow f SO flow o cycle C_i

Given a PNE flow f and a social optimum flow o, we consider the flow $o-f$. When G is series-parallel, $o-f$ contains only internally disjoint cycles (Fotakis, 2010). The set of cycles of $o-f$ is denoted by $\mathcal{C}$. For each cycle $C_i \in \mathcal{C}$, we denote define two paths C_i^- and C_i^+, where C_i^- contains edges where $f_e > o_e$ and C_i^+ contains edges where $f_e < o_e$.

Main result

Theorem 1. The price of anarchy of series-parallel network congestion games with affine delay functions is at most 2.

- The PoA of network congestion games with affine delay functions has a tight upper bound of 5/2 (Correa et al., 2019).
- On extension-parallel networks, a subclass of series-parallel networks, network congestion games with affine delay functions have a tight upper bound of 4/3 (Fotakis, 2010). However, this bound cannot be extended to series-parallel networks.

Proof of Theorem 1

We define $\Delta(f,o) := \sum_{C_i\in\mathcal{C}} \text{cost}_f(C_i^-) - \sum_{C_i\in\mathcal{C}} \text{cost}^+_f(C_i^+)$.

For affine delays, it holds:
$$\text{cost}(f) \leq \text{cost}(o) + \frac{1}{4}\text{cost}(f) + \Delta(f,o)$$

Main Lemma. In a series-parallel network congestion game with affine delay functions, we have $\Delta(f,o) \leq \frac{1}{4}\text{cost}(f)$.

Using the main lemma, we get that $\text{cost}(f) \leq 2\text{cost}(o)$, which implies PoA ≤ 2.

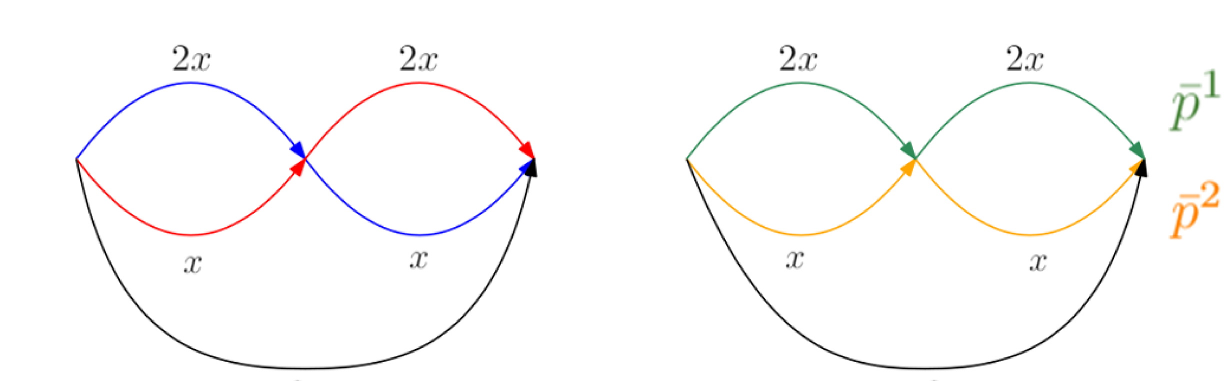
The Greedy Decomposition

Given a flow g and an edge costs vector $c \in \mathbb{R}^{|E|}$, where $c_e = d_e(g_e)$, we compute a greedy decomposition $\bar P(g) = \{\bar p^1,\dots,\bar p^N\}$ of g as follows:

- Set $g_1 = g$, let $E_1 \subseteq E$ be the edges with positive flows.

At each step:

- Compute the (s,t)-path $\bar p^i$ in (V,E_i) with highest cost w.r.t. c.
- Decrease the flow g_i by 1 on all the edges that belong to $\bar p^i$ to define g_{i+1} and E_{i+1}.



decomposition P^* of PNE flow f greedy decomposition $\bar P$ of f

Properties of the Greedy Decomposition

Let $P = \{p^1,\cdots .p^N\}$ be a decomposition of f and $x\in\mathbb{R}$. Define
$$R(P,x) := \sum_i^N \max\{0, \text{cost}_f(p^i) - x\}.$$
Let $\bar P = \bar P(f) = \{\bar p^1,\cdots .\bar p^N\}$ be a greedy decomposition of f.

1. $\text{cost}_f(\bar p^{i+1}) \geq \frac{1}{2}\sum_{j=1}^{i} \frac{\text{cost}_f(\bar p^j)}{i}$ for $i\in[N-1]$.
2. For any $x>0$, we have $R(\bar P, x) \geq R(P,x)$.

By these properties, we can show that when $\mathcal{C}$ contains only (s,t)-cycles:
$$\Delta(f,o) \leq R(\hat P, \frac{\text{cost}(f)}{N}) \leq R(\bar P, \frac{\text{cost}(f)}{N}) \leq \frac{1}{4}\text{cost}(f).$$
Where $\hat P$ is a decomposition containing all the paths C_i^-.

Extension to General Case

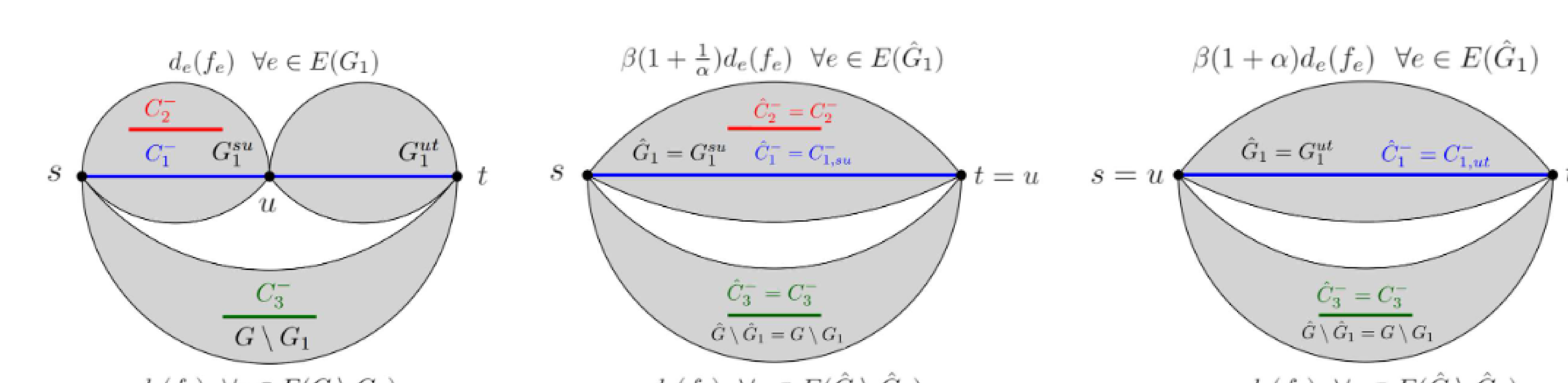
We show that $\Delta(f,o) \leq R(\bar P, \frac{\text{cost}(f)}{N})$ also holds for the case when there are some C_i are not from s to t.

- Define $\Delta(\mathcal{H},f) := \sum_{C_i\in\mathcal{H}} \text{cost}_f(C_i^-) - \sum_{C_i\in\mathcal{H}} \text{cost}^+_f(C_i^+)$. Note that this definition works for any set $\mathcal{H}$ of cycles. When $\mathcal{H}=\mathcal{C}$, we have $\Delta(\mathcal{C},f) = \Delta(f,o)$.
- Assume that G is composed in parallel by $G_1,\cdots,G_k$.

We repeatedly apply a network shrinking operations to construct a network $\hat G$, a PNE flow $\hat f$ and a set of cycles $\hat{\mathcal{C}}$, such that $\frac{\Delta(\hat{\mathcal{C}},\hat f)}{\text{cost}(\hat f)} \geq \frac{\Delta(\mathcal{C},f)}{\text{cost}(f)}$.

1. Pick a parallel component G_i who contains a non-(s,t) cycle.
2. G_i must be composed in series by two series-parallel subnetworks, we shrink one of them to get $\hat G$.
3. Scale the delay functions of $\hat G$ using parameters α and β.
4. Update $\hat{\mathcal{C}}$, $\hat f$ according to $\hat G$.

At the end, all the cycles in $\hat{\mathcal{C}}$ are from s to t. Then we can conclude:
$$\frac{\Delta(f,o)}{\text{cost}(f)} = \frac{\Delta(\mathcal{C},f)}{\text{cost}(f)} \leq \frac{\Delta(\hat{\mathcal{C}},\hat f)}{\text{cost}(\hat f)} \leq \frac{1}{4}.$$



(i) (ii) (iii)