Copositive Duality for Discrete Markets and Games

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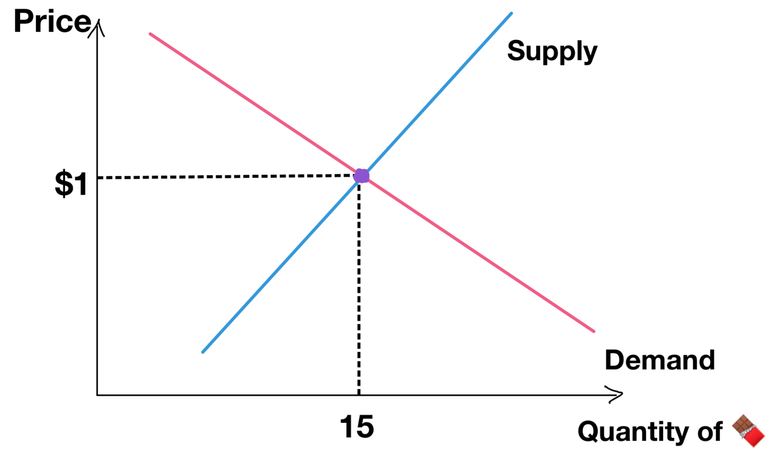
BACKGROUND

Economic Equilibrium

Examples:

→ Market equilibrium: supply = demand & individually rational



→ Nash equilibrium (NE): no deviation from equilibrium

Motivation

- Equilibrium for a convex economics problem is usually obtained by strong duality. E.g. shadow prices, Karush-Kuhn-Tucker (KKT) conditions for NE
- For nonconvex problems with discrete decisions, strong duality generally does not exist, which is a challenge

★ Our framework: mixed-binary quadratic programs (MBQPs) → reformulate to an equivalent convex (completely positive) program (Burer, 2009) → use strong duality of convex programs for discrete pricing and game problems.

Copositive Programming

- Copositive cone: $\mathcal{C} = \{X \in \mathcal{S} | y^\top X y \geq 0, \forall y \in \mathbb{R}^n_+\}$
- Completely positive cone: $\mathcal{C}^* = \{XX^\top | X \in \mathbb{R}^{n \times r}, X \geq 0\}$
- *CPP (completely positive program)*: optimize over $X \in \mathcal{C}^*$
- *COP (copositive program)*: optimize over $X \in \mathcal{C}$, dual of CPP
- Often solved by *semi-definite program (SDP) approximations*

MBQP → (Reformulation) → CPP → (Relaxation) → Optimization over $\mathcal{S}^+ \cap \mathcal{N}$

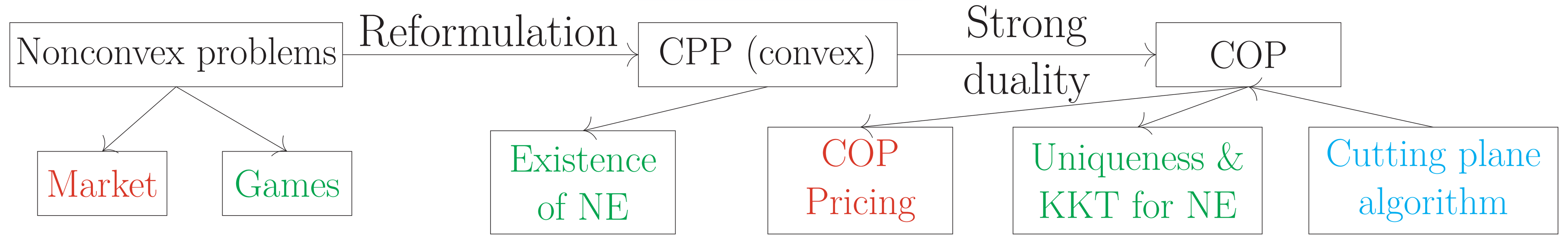
CPP → (Dualization) → COP → (Restriction) → Optimization over $\mathcal{S}^+ + \mathcal{N}$

Contributions

- A notion of duality for discrete problems
- A novel COP-based pricing scheme for nonconvex energy markets
- Theoretical results for mixed-binary quadratic games
- An exact cutting plane algorithm for mixed-integer COPs

FRAMEWORK & APPLICATIONS

Overview



Pricing in Energy Markets

Unit commitment (UC) problem: For each hour t and generator g decide

$$\begin{aligned}
\min\ & \sum_{g\in\mathcal{G}} \sum_{t\in\mathcal{T}} (c^p_g p_{gt} + c^u_g u_{gt}) \\
\text{s.t. } & \sum_{g\in\mathcal{G}} p_{gt} = d_t,\ \forall t \in \mathcal{T} && \rightarrow \text{Demand constraints } (\lambda) \\
& \mathbf{a}_{jgt}\mathbf{x} = b_{jgt},\ \forall j = [m], g \in \mathcal{G}, t \in \mathcal{T} && \rightarrow \text{Operational constraints } (\phi) \\
& z_{gt} \in \{0,1\},\ \forall g \in \mathcal{G}, t \in \mathcal{T}
\end{aligned}$$

- Variables:
  - p_{gt}: production level
  - u_{gt}: turn on decision
  - z_{gt}: on/off status
  - $\mathbf{x}^\top = (\mathbf{u}^\top, \mathbf{z}^\top, \mathbf{p}^\top)$

- Traditional pricing method: restricted pricing (RP)

Solve UC → Fix binary variables → Re-solve new linear program → Get dual prices (λ^*)

☹ Revenue generally does not recover operational costs

- Our pricing method: copositive dual pricing (CDP)

Reformulate UC as CPP → Obtain the dual COP → Solve the COP → Get dual prices $(\lambda^*, \phi^*, \Lambda^*, \Phi^*)$

– Thanks to strong duality of CPP:
☺ Generators: Total revenue = total costs (revenue neutrality)
☺ Individual rationality holds under certain conditions

Mixed-Binary Quadratic (MBQ) Games

- *n-person MBQ game*, each player solves an MBQP
- MBQ game ⇒ *completely positive (CP) game*
- NE of an MBQ game ⇔ NE of a CP game (under Slater's condition)

☺ Propose existence and uniqueness conditions of NE for MBQ games

☺ Obtain NE of an MBQ game via KKT conditions of the CP game

– Special case: only binary variables & equality constraints (e.g. bimatrix games)
→ KKT conditions can be reformulated to a single (mixed-integer) COP

ALGORITHM & NUMERICAL RESULTS

Cutting Plane for COPs

- COPs are often solved with SDP approximations
- We propose a novel cutting plane algorithm for mixed-integer COP problems:
  - *Step 1:* Solve a relaxed problem without the conic constraint $\Omega \in \mathcal{C}$
  - *Step 2:* Solve a MIP (Anstreicher, 2020) to separate the optimal $\hat{\Omega}$:

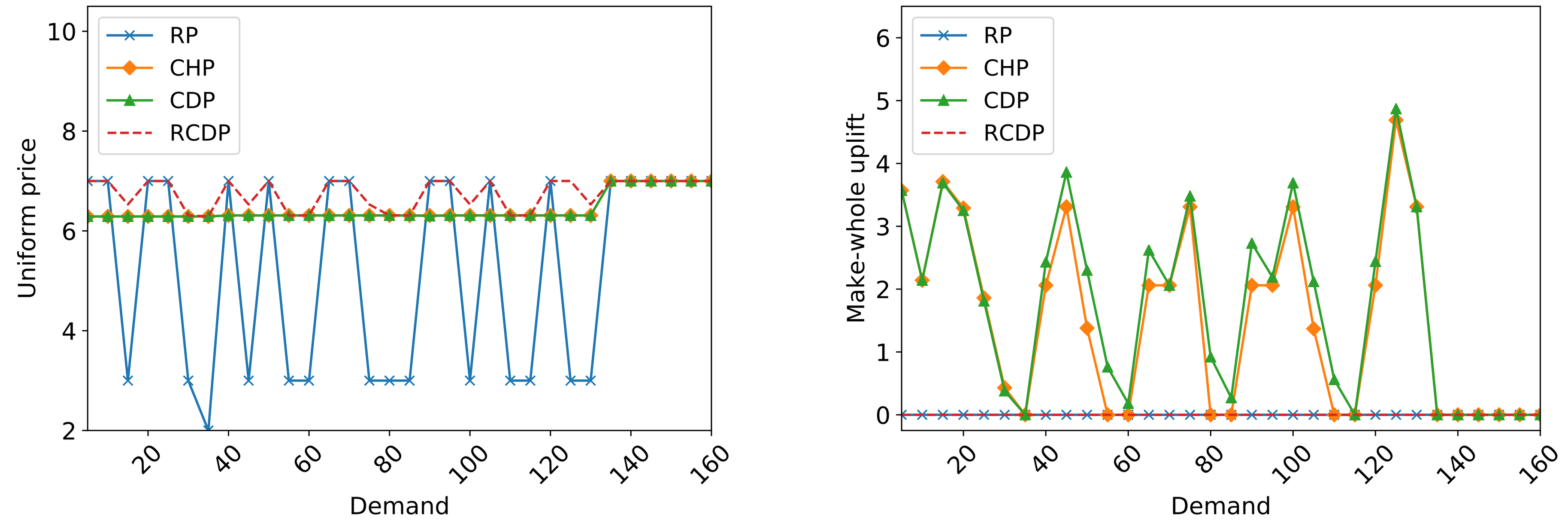
$$\begin{aligned}
\max\ & w \\
\text{s.t. } & \hat{\Omega}\mathbf{z} \leq -w\mathbf{1} + M(1-\mathbf{u}) \\
& \mathbf{1}^\top \mathbf{u} \geq q \\
& w \geq 0, \mathbf{z} \in [0,\mathbf{u}]^{n_c}, \mathbf{u} \in \{0,1\}^{n_c}
\end{aligned}$$

  - *Step 3:* If optimal $\bar{w} > 0$, add the cut $\bar{\mathbf{z}}^\top \Omega \bar{\mathbf{z}} \geq 0$

Max clique				Mosek			Cutting plane	
instance	$\lvert\mathcal{N}\rvert$	$\lvert\mathcal{E}\rvert$	ω	Obj	Gap(%)	Time(sec)	Time(sec)	#Iter
c-fat200-5	200	8473	58	60.35	3.89	606.33	12.19	2
hamming6-4	64	704	4	4	0	1.59	1.55	4
johnson16-2-4	120	5460	8	8	0	31.88	62.75	2
MANN_a9	45	918	16	17.48	8.47	0.45	547.62	2

Pricing Experiment: Scarf's Example

- *Scarf's example*: a classical nonconvex market example



- CHP (convex hull pricing, Hogan and Ring, 2003): Lagrangian dual prices
- RCDP (revenue-adequate CDP): *individual* revenue adequacy required in the dual COP

Game Experiment: Bimatrix Games

- Use *bimatrix games* for testing the KKT conditions
  – Converges pretty fast (slower than state-of-the-art bimatrix game algorithm)
  – Our method is more general, can be applied to other games

Size	Time (sec)	# Iterations
3 × 3	1.48	1
4 × 4	2.45	31
5 × 5	4.75	62