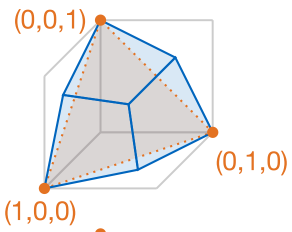
Deciding whether the {0,1/2}-closure of a given polyhedron coincides with the integer hull is strongly NP-hard

1 Integral {0,1/2}-closures: Two examples

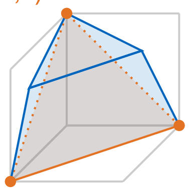
Consider a polyhedron $P = \{x \mid x \geq 0,\ Ax \leq b\}$ with A and b integral.

A {0,1/2}-**cut** for P is an inequality of the form $\lfloor uA \rfloor x \leq \lfloor ub \rfloor$ for a row vector u of multipliers 0 or 1/2 [Caprara, Fischetti 1996]. All {0,1/2}-cuts, together with the original system, define the {0,1/2}-**closure** of P.

$$x \geq 0\,,\quad \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 2 \\ 2 \\ -1 \end{pmatrix}$$



$$x \geq 0\,,\quad \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 2 & 1 \\ -1 & -1 & -1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 2 \\ 2 \\ -1 \end{pmatrix}$$



Can we easily tell from A and b whether the {0,1/2}-closure and the integer hull of P coincide?

The integer hull is equal to

$$\{x \geq 0 \mid x_1 + x_2 + x_3 = 1\}$$

in both examples. The inequality

$$x_1 + x_2 + x_3 \leq 1$$

is a {0,1/2}-cut only for the second polyhedron.

(Sum the highlighted inequalities with multipliers 1/2 each.)

2 Main result and related work

Given integral A and b, does the {0,1/2}-closure of $\{x \mid x \geq 0,\ Ax \leq b\}$ coincide with the integer hull?

Deciding this is strongly NP-hard, even for polytopes in the 0/1 cube.

Recognizing integrality of the Gomory-Chvátal closure of polyhedra is (weakly) NP-hard [Cornuéjols, Li 2018], even for polytopes in the 0/1 cube [Cornuéjols, Lee, Li 2020].

3 Reduction from Set Packing

Given a family of m subsets over a ground set of n items, is there a subfamily of k pairwise disjoint sets ($k \geq 2$) ?

m **blue** rows:

row i = (2,…,2) − incidence vector of set i

n columns

$$A = \begin{pmatrix} 1 & \cdots & 2 \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ -(2k-3) & \cdots & -(2k-3) \end{pmatrix},\quad b = \begin{pmatrix} 2 \\ \vdots \\ 2 \\ -(2k-3) \end{pmatrix}$$

The integer hull is equal to $\{x \geq 0 \mid x_1 + \cdots + x_n = 1\}$.

Which rows of A induce a {0,1/2}-cut that is equivalent to $x_1 + \cdots + x_n \leq 1$?

- Always select the **orange** row (unique inequality with odd right-hand side) along with at least k **blue** rows.
- Never select two **blue** rows with a 1 in the same column.

4 Byproducts

Strong NP-hardness of deciding whether …

- adding all {0,1/2}-cuts produces a totally dual integral system;
- the {0,1/2}-closure coincides with the Gomory-Chvátal closure.

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