Compact Formulations of Network Flow Problems

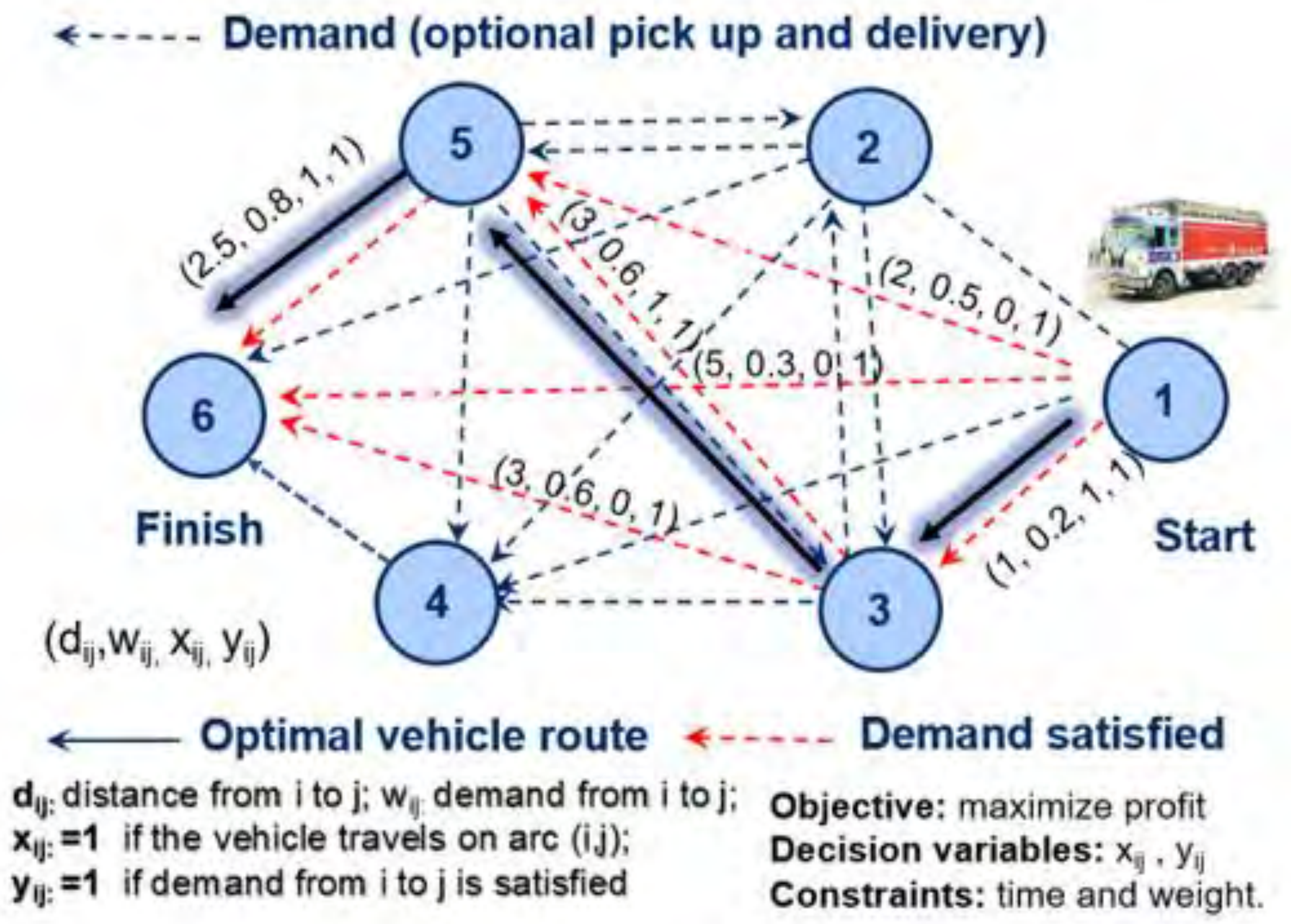
Yulan Bai[1] , Eli Olinick[1], Ronald Rardin[2] ,Yuanyuan Dong[1], Andrew Yu[3]

[1]Dept of EMIS, Southern Methodist University ,[2]Dept of IE, University of Arkansas , [3]Dept of ISE, University of Tennessee

Abstract

The Triples formulation is a compact formulation of multicommodity network flow problems that provides a different representation of flow than the traditional widely used node-arc and arc-path approaches. We have applied the Triples concept to the backhaul profit maximization problem (BPMP) and the fixed charge network flow problem (FCNF). Experimental analysis shows that the Triples formulations of BPMP and FCNF are more compact, stronger, and faster than the traditional node-arc models for most problem instances.

Backhaul Profit Maximization Problem



Flow: Node-Arc vs. Triples

- **Node-Arc Model**
  - –$z_{kl,ij}$ = 1 if demand from k to l is shipped on arc (i, j)
  - –Example solution (6 demands selected):
    - $z_{13,13}$ = 1, $z_{15,13}$ = $z_{15,35}$ = 1, $z_{16,13}$ = $z_{16,35}$ = $z_{16,56}$ =1
    - $z_{35,35}$ = 1, $z_{36,35}$ = $z_{36,56}$ = 1, $z_{56,56}$ = 1
  - –θ_{ij} = total flow (tons) on arc (i, j)
    - θ_{13} = w_{13} $z_{13,13}$ + w_{15} $z_{15,13}$ + w_{16} $z_{16,13}$ = 1 ton
    - θ_{35}=w_{15} $z_{15,35}$+ w_{16} $z_{16,35}$+w_{35} $z_{35,35}$+ w_{36} $z_{36,35}$= 2 tons
    - θ_{56} = w_{16} $z_{16,56}$ + w_{36} $z_{36,56}$ + w_{56} $z_{56,56}$ = 1.7 tons
  - –O(n^4) binary variables and O(n^3) constraints
- **Triples Model**
  - –u^k_{ij} = flow (tons) from i to j diverted through k: routed on arc (i, k) and a path from k to j
  - –Example solution diverts 3 of 6 selected demands
    - u^3_{15} = w_{15}, u^3_{16} = w_{16}, and u^5_{36} = w_{16}+ w_{36}
    - θ_{13} = w_{13} + u^3_{15} + u^3_{16} = 1 ton
    - θ_{35} = w_{35} + u^3_{15} + u^5_{36}= 2 tons
    - θ_{56} = w_{56} + u^5_{36} = 1.7 tons
  - –O(n^3) binary variables and O(n^2) constraints

BPMP: Node-Arc vs. Triples Models

Maximize $p\left[\sum_{(k,l)\in E} d_{kl} w_{kl} y_{kl}\right] - c\sum_{(i,j)\in E}\theta_{ij} d_{ij} - cv\sum_{(i,j)\in E} d_{ij} x_{ij}$

Subject to:

$\sum_{j\in V} z_{kl,kj} = y_{kl}\ \ (k,l)\in A$

$\sum_{i\in V} z_{kl,il} = y_{kl}\ \ (k,l)\in A$

$\sum_{i\in V,(i,a)\in A} z_{kl,ia} = \sum_{j\in V,(a,j)\in A} z_{kl,aj}\ \ (k,l)\in A,\ a\in V\backslash\{k,l\}$

$\sum_{(k,l)\in A} z_{kl,ij} \le (n^2-3n+3)x_{ij}\ \ (i,j)\in A$

$\sum_{(1,j)\in A} x_{1,j} = 1$

$\sum_{(i,n)\in A} x_{i,n} = 1$

$\sum_{i\in V\backslash\{k,n\}} x_{ik} = \sum_{j\in V\backslash\{1,k\}} x_{kj}\ \ k\in V\backslash\{1,n\}$

$\sum_{(i,j)\in A} d_{ij} x_{ij} \le D$

$\theta_{ij} = \sum_{(k,l)\in A} w_{kl} z_{kl,ij}\ \ (i,j)\in A$

$\theta_{ij} = w_{ij} y_{ij} + \sum_{(i,k,j)\in T} u^j_{ik} + \sum_{(k,j,i)\in T} u^i_{kj} - \sum_{(i,j,k)\in T} u^k_{ij},\ (i,j)\in A$

$\theta_{ij} \le Q,\ (i,j)\in A$

$\theta_{ij} \le Q x_{ij},\ (i,j)\in A$

$s_i - s_j + (n)x_{ij} \le n{-}1\ \ (i,j)\in A$

$x_{ij} = 0,1\ \ (i,j)\in A$

$y_{kl} = 0,1\ \ (k,l)\in A$

$u^k_{ij} \ge 0,\ (i,j,k)\in T$

$z_{kl,ij} = 0,1\ \ (k,l)\in A,\ (i,j)\in A$

Black: common for both models

Blue: unique to Node-arc model

Red: unique to Triples model

Restricted Triples Heuristic

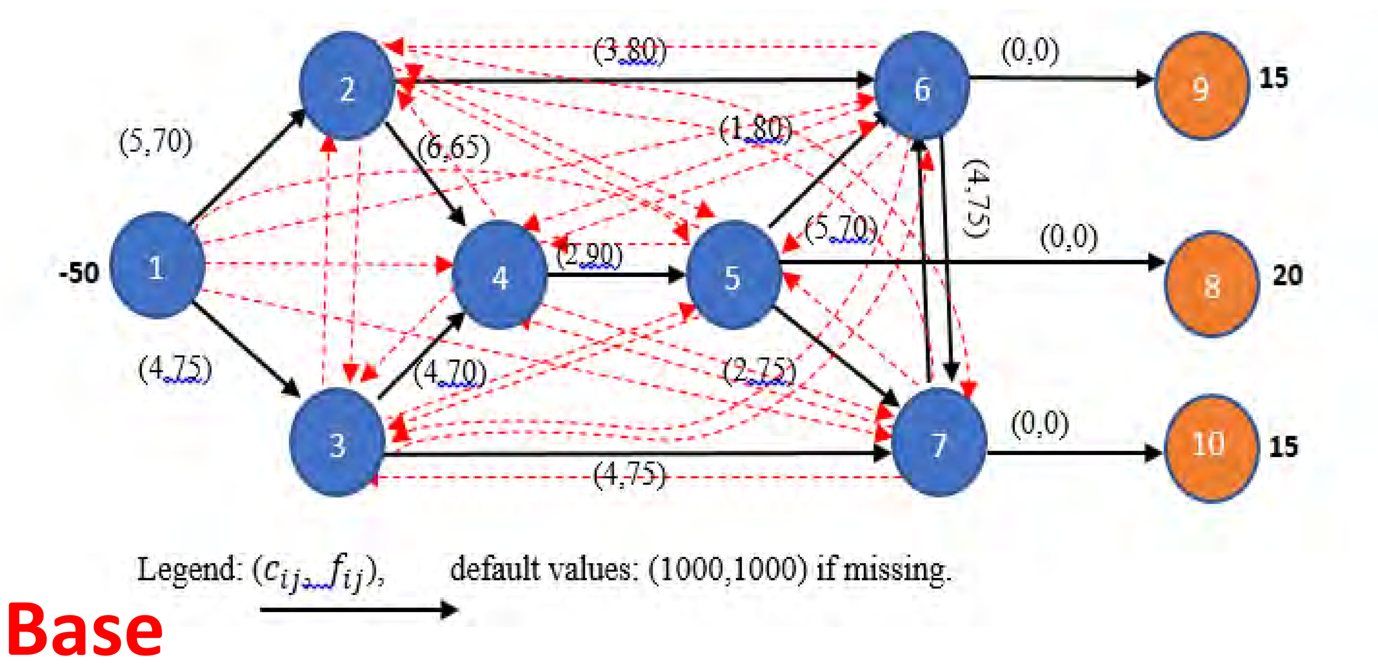
Algorithm: Restricted Triples Heuristic
Input: A BPMP instance
Output: A feasible triples solution

(1) $\hat{\mathcal{T}} \leftarrow \emptyset$
(2) **For** $\forall(i,j,k)\in\mathcal{T}$
(3) $\rho^k_{ij} \leftarrow pd_{ij}w_{ij} - c(d_{ik}+d_{kj})(v+w_{ij})$
(4) **If** $w_{ij}+w_{ik}\le Q$ **Then** $\rho^k_{ij}\leftarrow\rho^k_{ij}+(p-c)d_{ik}w_{ik}$
(5) **If** $w_{ij}+w_{kj}\le Q$ **Then** $\rho^k_{ij}\leftarrow\rho^k_{ij}+(p-c)d_{kj}w_{kj}$
(6) **If** $\rho^k_{ij}\ge 0$ **Then** $\hat{\mathcal{T}}\leftarrow\hat{\mathcal{T}}\cup\{(i,j,k)\}$
(7) **End For**
(8) Solve enhanced triples formulation with $\hat{\mathcal{T}}$ (i.e., fix $u^k_{ij}=0$ for $(i,j,k)\in\mathcal{T}\setminus\hat{\mathcal{T}}$)
(9) $\mathcal{A}_x \leftarrow \{(i,j)\in\mathcal{A} : x_{ij}=1\}$
(10) $\mathcal{R}_y \leftarrow \{(i,j)\in\mathcal{R} : y_{ij}=1\}$
(11) Solve enhanced triples formulation with $\mathcal{T}$ subject to
$x_{ij}=1\quad \forall(i,j)\in A_x,$
$y_{ij}=1\quad \forall(i,j)\in R_y.$
(12) **Return** Triples solution (x,y,u,s)

Empirical CPU Comparison

	Average CPU (seconds)		
Nodes	Node Arc	Triples	Speed Up
10	14.4	2.4	1.5
20	13,644	20	758
30	N/A	480	NA

Fixed Charge Network Flow Problem



Base

Min $\sum_{(i,j)\in A} c_{ij}\,x_{ij} + \sum_{(i,j)\in A} f_{ij} y_{ij}$

$\sum_{(i,k)\in A} x_{ik} - \sum_{(k,j)\in A} x_{kj} = b_k,\ \ k\in V$

$x_{ij} \le (\sum_{d\in D} b_d) y_{ij},\ (i,j)\in A$

$x_{ij} \ge 0,\ (i,j)\in A$

$y_{ij} \in \{0,1\}$

MCE

Min $\sum_{(i,j)\in A} x_{ij} c_{ij} + \sum_{(i,j)\in A} f_{ij} y_{ij}$

$\sum_{(i,k)\in A} w^d_{ik} - \sum_{(k,j)\in A} w^d_{kj} = \begin{cases} b_d & k = d\in D \\ 0, & k\in V\backslash 1,\ d\in D,\ d\ne k \\ -b_d & k = 1,\ d\in D \end{cases}$

$x_{ij} \le (\sum_{d\in D} b_d) y_{ij},\ (i,j)\in A$

$x_{ij} \ge \sum_{d\in D} w^d_{ij}\ (i,j)\in A$

$w^d_{ij} \le b_d y_{ij},\ (i,j)\in A,\ d\in D$

$w^d_{ij} \ge 0,\ (i,j)\in A,\ d\in D$

$y_{ij} \in \{0,1\}$

Triples

Minimize $\sum_{(i,j)\in A} x_{ij} c_{ij} + \sum_{(i,j)\in A} f_{ij} y_{ij}$

$x_{ij} = d_{ij} + \sum_{(i,k,j)\in T} u^j_{ik} + \sum_{(k,j,i)\in T} u^i_{kj} - \sum_{(i,j,k)\in T} u^k_{ij}\ \ (i,\ j) : i\in V, j\in V\backslash\{i\}$

$T:\{(i,j,k):\ i\in V, j\in V\backslash\{i\}, k\in V\backslash\{i,j\}\}$

$x_{ij} \le s_{ij} * y_{ij},\ (i,\ j) : i\in V, j\in V\backslash\{i\}$

$u^k_{ij} \ge 0\ \ \forall\,(i,j,k)\in T$

$y_{ij} \in \{0,1\}\ \ \forall(i,j)\in A$

Experimental Comparison

Node	MCE vs. Base				Triples vs. Base			
	CPU speed up	ticks speed up	Real time speedup	LP lower improvement	CPU speed up	ticks speedup	Real time speedup	LP lower improvement
20	6.45	1.34	1.91	36.8%	6.41	1.34	1.94	36.8%
30	6.15	1.12	1.56	26.6%	5.85	1.12	1.51	26.6%
100	7.02	1.44	1.62	89.1%	6.79	1.44	1.59	89.1%
500	1.41	16.29	0.12	320.1%	1.38	16.29	0.11	320.1%
Average	5.26	5.05	1.30	118.1%	5.10	5.05	1.29	118.1%

Conclusions and Future Work

- The Triples model is more compact and faster to solve than node-arc model for BPMP.
- The Triples model is at least as good as the traditional advanced MCE formulation for FCNF.
- In the future, we will explore singleton-selection strategies to further accelerate our solution approach.