EE-2025 Fall-2001

Lecture 19
Fourier Transform Properties
9-Nov-01

Info: Web-CT, Lab, HW

- Calendar:
  - Quiz #3 is Monday, 19-Nov
  - One page hand-written notes
  - Calculator is OK
- **REVIEW: Sunday (18-Nov) at 6:30pm**
  - **ECE Auditorium**
- Prob Set #10 is due NEXT WEEK
  - Solution will be posted Friday

Strategy for using the FT

- Develop a set of known Fourier transform pairs.
- Develop a set of "theorems" or properties of the Fourier transform.
- Develop skill in formulating the problem in either the time-domain or the frequency-domain, *which ever leads to the simplest solution*.

LECTURE

READING ASSIGNMENTS

- This Lecture:
  - Chapter 12, pp. 1093-1097, 1102-1118
  - **Tables on pp. 1117 & 1118**
- Other Reading:
  - Recitation: All of Chapter 12
  - Next Lecture: Chapter 13, pp. 1143-1156

LECTURE OBJECTIVES

- The Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- More examples of Fourier transform pairs
- Basic properties of Fourier transforms
  - **Convolution** property
  - **Multiplication** property

Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Fourier Synthesis (Inverse Transform)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Fourier Analysis (Forward Transform)

$$\text{Time-domain} \Leftrightarrow \text{Frequency-domain}$$

$$x(t) \Leftrightarrow X(j\omega)$$

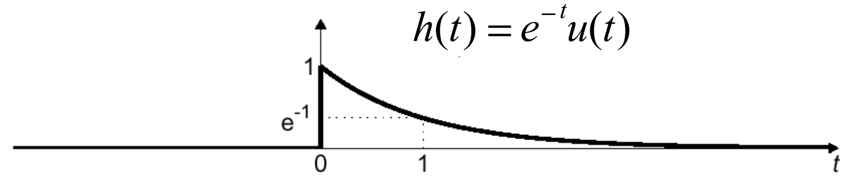
WHY use the Fourier transform?

- Manipulate the **"Frequency Spectrum"**
- Analog Communication Systems
  - AM: Amplitude Modulation; FM
  - What are the "**Building Blocks**" ?
    - **Abstract Layer**, not implementation
- Ideal Filters: mostly BPFs
- Frequency Shifters
  - Modulators, or Multipliers: x(t)p(t)

Frequency Response

- Fourier Transform of h(t) is the Frequency Response

$$h(t) = e^{-t}u(t)$$



$$h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1+j\omega}$$

Table of Fourier Transforms

$$x(t) = e^{-at}u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

Example 5: $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

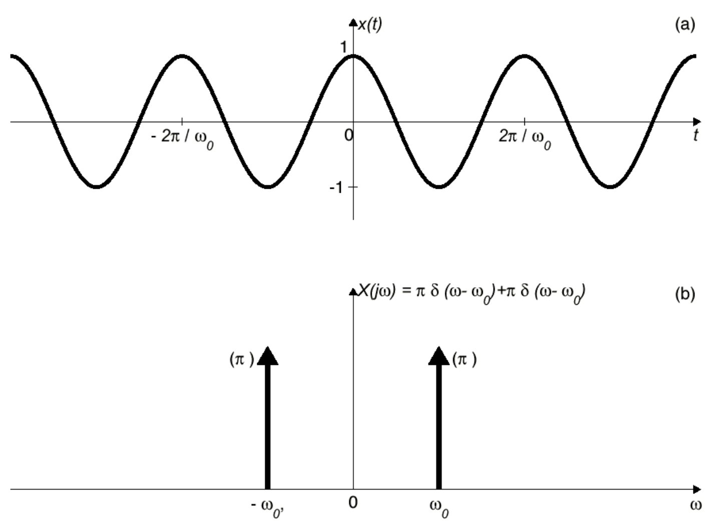
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\,d\omega = e^{j\omega_0 t}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)$$

$$x(t) = \cos(\omega_0 t) \Leftrightarrow X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$x(t) = \cos(\omega_0 t) \Leftrightarrow X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



Fourier Transform of a General Periodic Signal

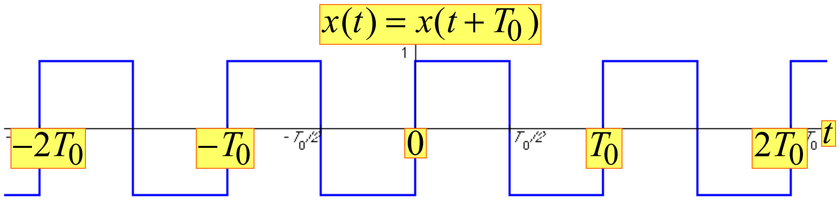
- If *x(t)* is periodic with period T_0,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt$$

Therefore, since $e^{jk\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - k\omega_0)$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

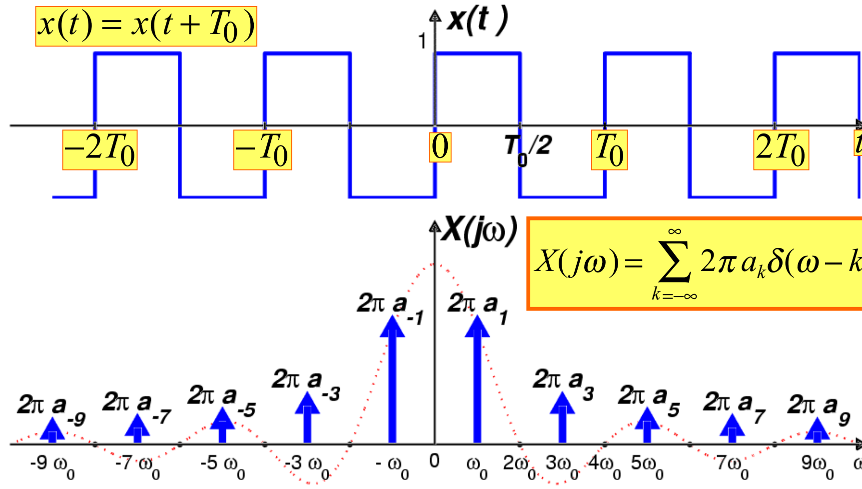
Square Wave Signal



$$a_k = \frac{1}{T_0}\int_0^{T_0/2} (1)e^{-j\omega_0 kt}dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} (-1)e^{-j\omega_0 kt}dt$$

$$a_k = \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_0^{T_0/2} - \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_{T_0/2}^{T_0} = \frac{1-e^{-j\pi k}}{j\pi k}$$

Square Wave Fourier Transform



$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Table of Easy FT Properties

Linearity Property

$$ax_1(t) + bx_2(t) \Leftrightarrow aX_1(j\omega) + bX_2(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Scaling

$$x(at) \Leftrightarrow \tfrac{1}{|a|} X(j(\tfrac{\omega}{a}))$$

Scaling Property

$$x(at) \Leftrightarrow \tfrac{1}{|a|} X(j(\tfrac{\omega}{a}))$$

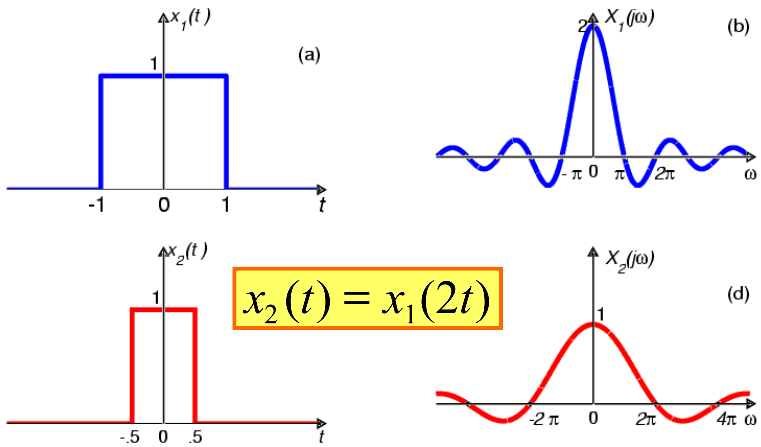
$$\int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau/a)}(\tfrac{1}{|a|})d\tau$$

$$= \tfrac{1}{|a|} X(j\tfrac{\omega}{a})$$

$x(2t)$ shrinks; $\tfrac{1}{2} X(j\tfrac{\omega}{2})$ expands

Scaling Property

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\tfrac{\omega}{a}))$$



$$x_2(t) = x_1(2t)$$

Uncertainty Principle

- Try to make x(t) shorter
  - Then X(jω) will get wider
  - Narrow pulses have wide bandwidth
- Try to make X(jω) narrower
  - Then x(t) will have longer duration
- **Cannot simultaneously reduce time duration and bandwidth**

Significant FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

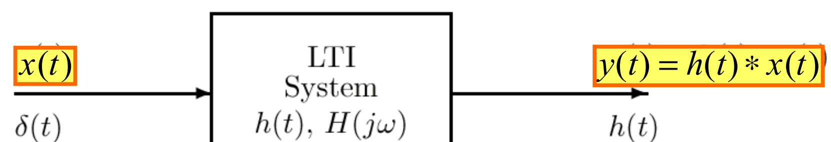
$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

Convolution Property



$X(j\omega)$ $\qquad Y(j\omega) = H(j\omega)X(j\omega)$

- Convolution in the time-domain

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

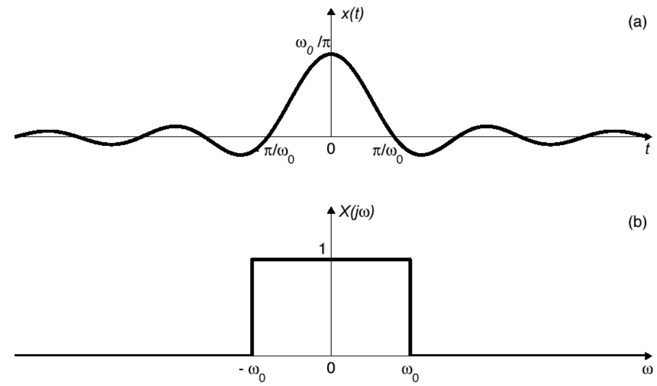
corresponds to **MULTIPLICATION** in the frequency-domain $\quad Y(j\omega) = H(j\omega)X(j\omega)$

Convolution Example

- Bandlimited **Input** Signal
  - "sinc" function
- Ideal LPF (Lowpass Filter)
  - **h(t)** is a "sinc"
- **Output** is Bandlimited
  - Convolve "sincs"

Ideally Bandlimited Signal

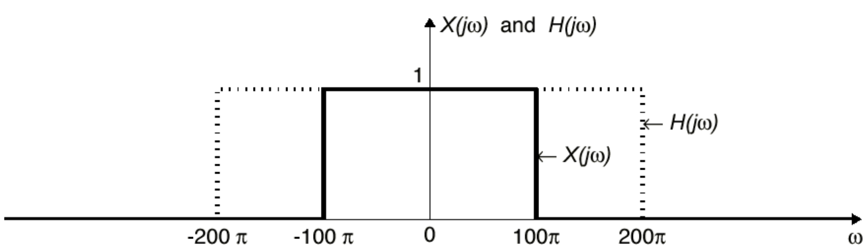
$$x(t) = \frac{\sin(100\pi t)}{(\pi t)} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < 100\pi \\ 0 & |\omega| > 100\pi \end{cases}$$



Convolution Example

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$\frac{\sin(100\pi t)}{\pi t} * \frac{\sin(200\pi t)}{\pi t} = \frac{\sin(100\pi t)}{\pi t}$$



Cosine Input to LTI System

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$= H(j\omega)[\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)]$$

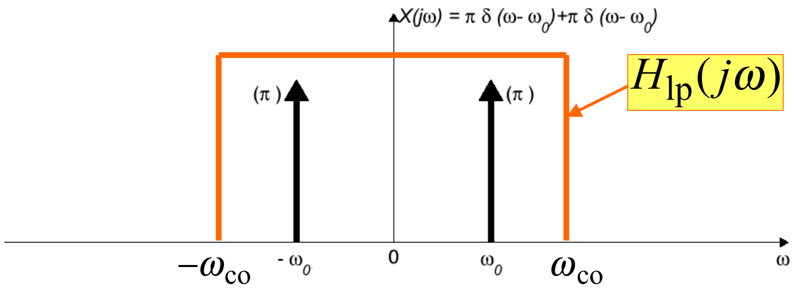
$$= H(j\omega_0)\pi\delta(\omega-\omega_0) + H(-j\omega_0)\pi\delta(\omega+\omega_0)$$

$$y(t) = H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H(-j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t}$$

$$= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H^{*}(j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t}$$

$$= |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0))$$

Ideal Lowpass Filter

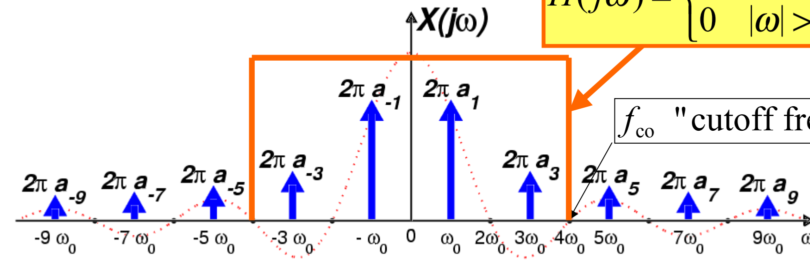


$$y(t) = x(t) \qquad \text{if } \omega_0 < \omega_{co}$$

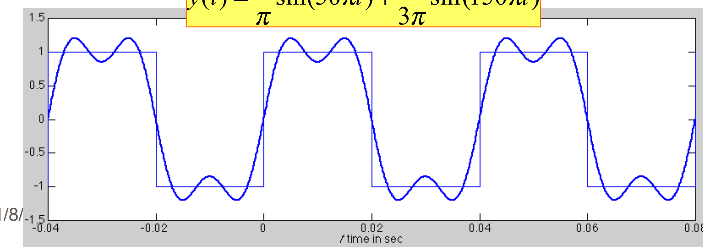
$$y(t) = 0 \qquad \text{if } \omega_0 > \omega_{co}$$

Ideal Lowpass Filter

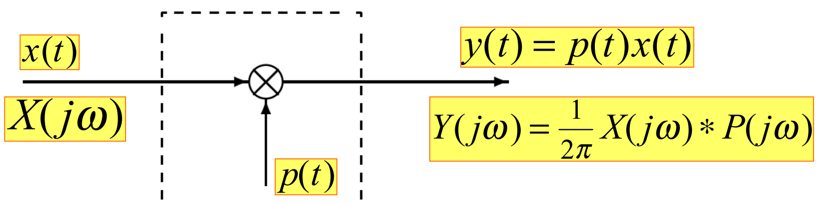
$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$



$$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t)$$



Signal Multiplier (Modulator)



- Multiplication in the time-domain corresponds to **convolution** in the frequency-domain.

$$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))d\theta$$

Differentiation Property

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega)X(j\omega)e^{j\omega t}d\omega$$

Multiply by jω

$$\frac{d}{dt}\left(e^{-at}u(t)\right) = -ae^{-at}u(t) + e^{-at}\delta(t)$$

$$= \delta(t) - ae^{-at}u(t)$$

$$\Leftrightarrow \frac{j\omega}{a+j\omega}$$