GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 2025 Fall 2001
Problem Set #4

Assigned: 14-September-01
Due Date: Week of 25-September-01

- Reading: In *DSP First*, all of Chapter 3 (revised) on *Spectrum Representation.*
- **Please check the "Bulletin Board" often. All official course announcements are posted there.**
- **ALL** of the **STARRED** problems need to be turned in for grading. A solution will be posted to the web. Some problems have solutions similar to those found on the CD-ROM or in the class archives.

Your homework is always due in recitation at the beginning of class. After the beginning of your assigned recitation time, the homework is considered late and will be given a zero.

PROBLEM 4.1*:

Let $x(t)$ be the signal

$$x(t) = [10 + 5\cos(30,000\pi t - \pi/2)]\cos(100,000\pi t + \pi/2).$$

(a) Use Euler's relation to expand $x(t)$ as a sum of complex exponential signals and show that it can be expressed in the Fourier series form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

(b) Determine the fundamental frequency ω_0 of this signal.

(c) What is the "DC value" of this signal?

(d) Determine all of the non-zero coefficients a_k of this signal and plot the spectrum of this signal. **Note carefully that you should be able to do this without evaluating any integrals.**

PROBLEM 4.2*:
A periodic signal $x(t)$ is described over one period $0 \le t \le T_0$ by the equation

$$x(t) = \begin{cases} \frac{2t}{T_0} & 0 \le t < T_0/2 \\ \frac{2(T_0 - t)}{T_0} & T_0/2 \le t \le T_0. \end{cases}$$

We have seen that such a periodic signal can be represented by the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{where} \quad a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

(a) Sketch the periodic function $x(t)$ for $-T_0 < t < 2T_0$.

(b) Determine a_0, the D.C. coefficient for the Fourier series.

PROBLEM 4.3*:
This is a continuation of the previous problem. Use the same signal $x(t)$ as in Problem 4.2.

(a) Use the Fourier $\boxed{\textit{analysis}}$ integral[1] (for $k \ne 0$)

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

to determine a general formula[2] for the Fourier Series coefficients a_k. Your final result for a_k should depend on k. Simplify your formulas by using $\omega_0 = \frac{2\pi}{T_0}$.

(b) Use the Fourier Series coefficients to sketch the spectrum of $x(t)$. Include *only* those frequency components corresponding to $k = 0, \pm1, \pm2, \pm3$. Label each component with its frequency and its complex amplitude (i.e., numerical values of magnitude and phase).

[1]The Fourier integral can be done over any period of the signal.
[2]The Fourier integral requires integration by parts—an opportunity to use your calculus skills.

PROBLEM 4.4:
We have seen that a periodic signal $x(t)$ can be represented by the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \tag{1}$$

It turns out that we can transform many operations on the signal into corresponding operations on the Fourier coefficients a_k. For example, suppose that we want to consider a new periodic signal $y(t) = \frac{dx(t)}{dt}$. What would the Fourier coefficients be for $y(t)$? To see this, we simply need to differentiate the Fourier series representation; i.e.,

$$y(t) = \frac{dx(t)}{dt} = \frac{d}{dt}\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right] = \sum_{k=-\infty}^{\infty} a_k \frac{d}{dt}\left[e^{jk\omega_0 t}\right] = \sum_{k=-\infty}^{\infty} a_k \left[(jk\omega_0) e^{jk\omega_0 t}\right]. \tag{2}$$

Thus, we see that $y(t)$ is also in the Fourier series form

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}, \quad \text{where} \quad b_k = (jk\omega_0) a_k,$$

but in this case the Fourier series coefficients are related to the Fourier series coefficients of $x(t)$ by $b_k = (jk\omega_0)a_k$. This is a nice result because it allows us to find the Fourier coefficients *without* actually doing the differentiation of $x(t)$ and *without* doing any tedious evaluation of integrals to obtain the Fourier coefficients b_k. It is a *general* result that holds for every periodic signal and its derivative.

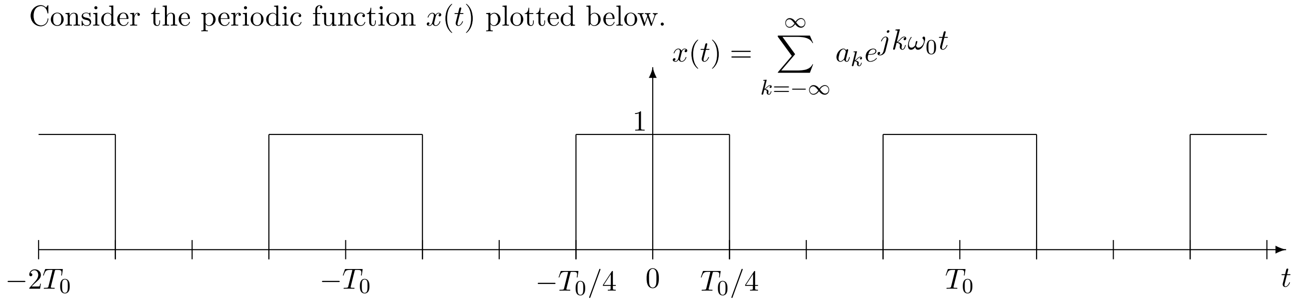
We can use this style of manipulation to obtain some other useful results for Fourier series. In each case below, use Equation (1) as the starting point and the given definition for $y(t)$ to express $y(t)$ as a Fourier series and then manipulate the equation so that you can pick off an expression for the Fourier coefficients b_k as a function of the original coefficients a_k.

(a) Suppose that $y(t) = Ax(t)$ where A is a real number; i.e., $y(t)$ is just a scaled version of $x(t)$. Show that the Fourier coefficients for $y(t)$ are $b_k = Aa_k$.

(b) Suppose that $y(t) = x(t - t_d)$ where t_d is a real number; i.e., $y(t)$ is just a delayed version of $x(t)$. Show that the Fourier coefficients for $y(t)$ in this case are $b_k = a_k e^{-jk\omega_0 t_d}$.

PROBLEM 4.5*:

Consider the periodic function $x(t)$ plotted below.



(a) Find the "DC" value a_0 and the other Fourier coefficients a_k for $k \neq 0$ in the Fourier series representation of $x(t)$.

(b) Sketch the waveform of the signal $y(t) = 2x(t)+1$ and write down the Fourier series coefficients b_0 and b_k for $k \neq 0$ for the periodic signal $y(t)$. You should be able to do this without evaluating any integrals.

(c) Sketch the waveform of the signal $z(t) = x(t - \frac{T_0}{2})$. Write down the Fourier series coefficients c_0 and c_k for $k \neq 0$ for the periodic signal $z(t)$. Here too you should be able to do this without evaluating and integrals. (*Hint:* The results of the previous problem might be helpful.)

PROBLEM 4.6*:

A linear-FM "chirp" signal is one that sweeps in frequency from $\omega_1 = 2\pi f_1$ to $\omega_2 = 2\pi f_2$ as time goes from $t = 0$ to $t = T_2$. We can define the *instantaneous frequency* of the chirp as the derivative of the "angle" of the sinusoid:

$$x(t) = A\cos(\alpha t^2 + \beta t + \phi) \tag{3}$$

where the cosine function operates on a time-varying angle argument

$$\psi(t) = \alpha t^2 + \beta t + \phi$$

The derivative of the angle argument $\psi(t)$ is the *instantaneous frequency,* which is also the audible frequency heard from the chirp.[3]

$$\omega_i(t) = \frac{d}{dt}\psi(t) \qquad \text{radians/sec} \tag{4}$$

(a) For the "chirp" signal

$$x(t) = \Re e\left\{e^{j2\pi(-100t^2 + 600t + 66)}\right\}$$

derive a formula for the *instantaneous* frequency versus time.

(b) For the signal in part (a), make a plot of the *instantaneous* frequency (in Hz) versus time over the range $0 \leq t \leq 2$ sec.

(c) Determine the formula for a signal $x(t)$ that sweeps from $f_1 = 5000$ Hz at $T_1 = 0$ secs. to $f_2 = 1000$Hz at $T_2 = 2$ secs.

(d) Sketch the time-frequency diagram showing the instantaneous frequency versus time for the signal in part (c).

[3] The instantaneous frequency is the frequency heard by the human ear when the chirp rate is relatively slow. There are cases of FM where the audible signal is quite different, but these happen when the chirp rate is very high.