SOLUTIONS P.S. # 11

Problem 11.1

a) $$X(j\omega) = j\cdot 20\pi\, \delta(\omega-50\pi) - j\, 20\pi\, \delta(\omega+50\pi)$$
$$= j\cdot 10\,[2\pi\, \delta(\omega-50\pi)] - j\cdot 10\,[2\pi\, \delta(\omega+50\pi)]$$

$$x(t) = j\cdot 10\, e^{j50\pi t} - j\cdot 10\, e^{-j50\pi t}$$
$$= j\cdot 10\cdot \frac{e^{j50\pi t} - e^{-j50\pi t}}{2j}\cdot 2j$$
$$= -20\cdot \sin 50\pi t$$

b) $x(t) = \sin 4\pi t \cdot \sin 50\pi t = \frac{1}{2}\cos 46\pi t - \frac{1}{2}\cos 54\pi t$

Noting that:
$$x(t) = \cos \omega_0 t \longleftrightarrow X(j\omega) = \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

$$X(j\omega) = \frac{\pi}{2}\left[\delta(\omega-46\pi) + \delta(\omega+46\pi)\right] - \frac{\pi}{2}\left[\delta(\omega-54\pi) + \delta(\omega+54\pi)\right]$$

Alternatively: $x_1(t) = \sin 50\pi t \longleftrightarrow X_1(j\omega) = \frac{\pi}{j}\left[\delta(\omega-50\pi) - \delta(\omega+50\pi)\right]$

$$x(t) = \sin 4\pi t\, \sin 50\pi t$$
$$= x_1(t)\cdot \sin 4\pi t \longleftrightarrow X(j\omega) = \frac{1}{2j}\left[X_1(j(\omega-4\pi)) - X_1(j(\omega+4\pi))\right]$$

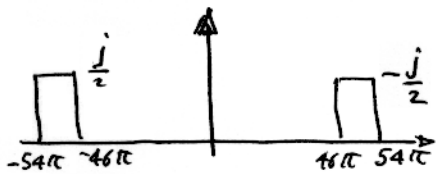
So: $$X(j\omega) = -\frac{\pi}{2}\left[\delta(\omega-54\pi) - \delta(\omega+46\pi) - \delta(\omega-46\pi) + \delta(\omega+54\pi)\right]$$

c) $x(t) = \frac{\sin 4\pi t}{\pi t}\sin 50\pi t$

Noting that $x_1(t) = \frac{\sin 4\pi t}{\pi t} \longleftrightarrow X_1(j\omega) = \begin{cases} 1 & |\omega| < 4\pi \\ 0 & |\omega| > 4\pi \end{cases}$

$$X(j\omega) = -\frac{1}{2j}\left\{\begin{matrix} 1, & \omega \in (-54\pi, -46\pi) \\ 0, & \text{otherwise}\end{matrix}\right\} + \frac{1}{2j}\left\{\begin{matrix} 1; & \omega \in (46\pi, 54\pi) \\ 0; & \text{otherwise}\end{matrix}\right\}$$

$$X(j\omega) = \begin{cases} -\frac{j}{2}, & \omega \in (46\pi, 54\pi) \\ +\frac{j}{2}, & \omega \in (-54\pi, -46\pi) \\ 0, & \text{else}\end{cases}$$



Problem 11.2

a) $Y(j\omega) = H(j\omega)\,X(j\omega)$

$$= \frac{2\pi}{2}\,\delta(\omega) \quad \longleftrightarrow \quad y(t) = \frac{1}{2}$$

b) $Y(j\omega) = 500\, e^{-j2\omega/3}\, X(j\omega)$

$\updownarrow$ delay property

$$y(t) = 500\, x\left(t - \frac{2}{3}\right)$$

c) $$Y(j\omega) = \left[\begin{cases} e^{-j2\omega/3} & |\omega| < 3\omega_0/2 \\ 0 & |\omega| > 3\omega_0/2 \end{cases}\right] X(j\omega)$$

$$= \underbrace{e^{-j2\omega/3}}_{\text{delay prop.}} \cdot \underbrace{\left[\begin{cases} 1 & |\omega| < 3\omega_0/2 \\ 0 & |\omega| > 3\omega_0/2 \end{cases}\right]}_{\text{this filters out all frequencies } \omega > \frac{3\omega_0}{2}} X(j\omega)$$

(∴ only DC and freq. ω_0 are left)

$$y(t) = \text{delayed } \left(+\tfrac{2}{3}\right) \text{ version of } \left\{ \frac{1}{2} + \frac{2}{\pi}\cos\omega_0 t \right\}$$

$$= \frac{1}{2} + \frac{2}{\pi}\cos\omega_0\left(t - \frac{2}{3}\right)$$

d) $$Y(j\omega) = \underbrace{\left[\begin{cases} 0 & |\omega| < \omega_0/2 \\ 1 & |\omega| > \omega_0/2 \end{cases}\right]}_{\text{filters out (only) the DC component of } x(t)} X(j\omega)$$

$$y(t) = x(t) - \frac{1}{2}$$

e) $\left.\begin{array}{l}\text{Passes the DC without attenuation} \\ \text{Attenuates } \left(\frac{2}{\pi}\cos\omega_0 t\right) \text{ by factor } 1/2 \\ \text{Blocks the rest}\end{array}\right\}$ $y(t) = \frac{1}{2} + \frac{1}{\pi}\cos\omega_0 t$

f) Passes ONLY the $\frac{2}{\pi}\cos\omega_0 t$ component : $y(t) = \frac{2}{\pi}\cos\omega_0 t$

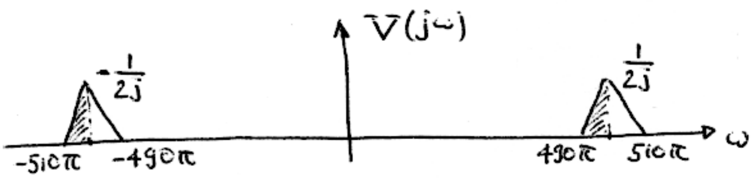
Problem 11.3

a) $H(j\omega) = \begin{cases} 1 & 500\pi < |\omega| < 510\pi \\ 0 & \text{otherwise} \end{cases}$

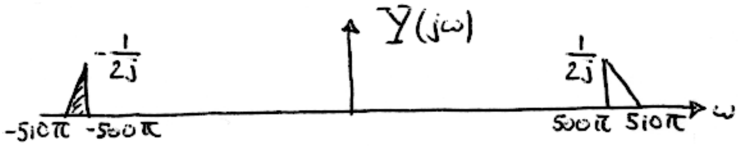
b) $v(t) = x(t)\sin(500\pi t)$

$$= \frac{1}{2j}\, x(t)\, e^{j500t\pi} - \frac{1}{2j}\, x(t)\, e^{-j500t\pi}$$

$$V(j\omega) = \frac{1}{2j} X\big(j(\omega - 500\pi)\big) - \frac{1}{2j} X\big(j(\omega + 500\pi)\big)$$

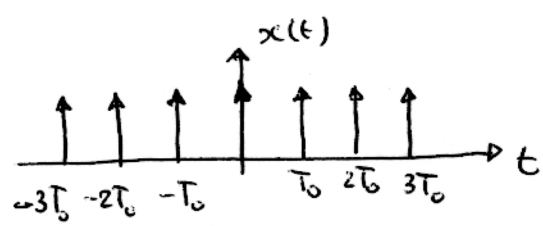


c)



Problem 11.4

a)



b) Period $T_0 = 10$, fundamental frequency $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{10} = \frac{\pi}{5}$

c) $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$; $a_k = \frac{1}{T_0} \int_{\text{period}} x(t) e^{-jk\omega_0 t} dt$

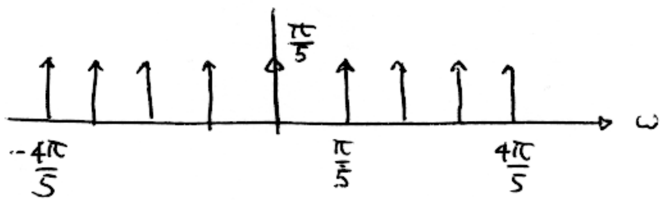
$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0} = \frac{1}{10}$$

d) From c)

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{10} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{10} e^{jk\frac{\pi t}{5}}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{10}\, \delta\left(\omega - k\frac{\pi}{5}\right)$$

For $|\omega| \leq 4\omega_0 = \frac{4\pi}{5}$



e) $H(j\omega)$ blocks all frequencies $> \pi/T_0 = \pi/10$
and passes the other frequency components with a delay of 4 units (of time)
Since $\pi/10 < \pi/5$, only the DC term will be passed.

$$\Rightarrow y(t) = \frac{\pi}{5}$$

f) As in e) but now all frequencies $> 3\pi/T_0 = 3\pi/10$ are blocked. (This leaves (and delays) the components at $\omega = 0$, $\omega = \pm\frac{\pi}{5}$.

$$\Rightarrow y(t) = \frac{\pi}{5} + \frac{\pi}{5} e^{j\frac{\pi}{5}(t-4)} + \frac{\pi}{5} e^{-j\frac{\pi}{5}(t-4)}$$

$$= \frac{\pi}{5} + \frac{2\pi}{5} \cos \frac{\pi}{5}(t-4)$$