ECE 2025 HOMEWORK 10 SOLUTIONS   DUE WEEK OF 3 APRIL 2000

NOTATION: FORWARD FOURIER TRANSFORM

$$x(t) \longleftrightarrow X(j\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

REVERSE FOURIER TRANSFORM

$$X(j\omega) \longleftrightarrow x(t) = \mathcal{F}^{-1}\{X(j\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{+j\omega t} \, d\omega$$

10.1a) $x(t) = 2 \cdot \delta(t-2) \cdot e^{t}$

$$X(j\omega) = 2 * \mathcal{F}\{\delta(t-2)\} * \mathcal{F}\{e^{t}\}$$

See Tables 12.1 + 12.2 p 1240 + 1241

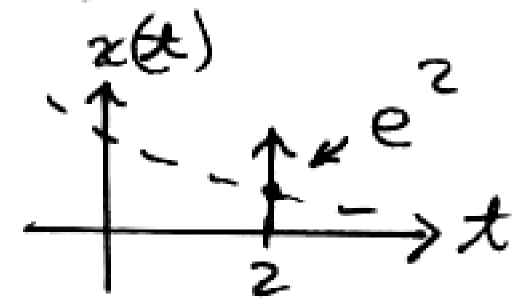
DO NOT CONFUSE WITH $e^{-kt}u(t)$

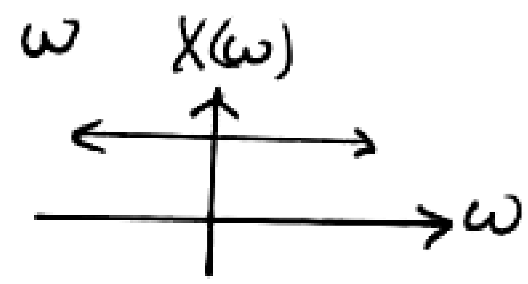
GOOD TRY BUT...

e^{t}, $u(t)$ } BOTH ARE UNBOUNDED $\Rightarrow$ NO FOURIER TRANSFORM (SEE EQN 12.3.1)

$$X(j\omega) = \mathcal{F}\{2 \cdot \delta(t-2) \cdot e^{t} = 2e^{2}\delta(t-2)\} = 2e^{2} e^{-j\omega 2} \simeq 14.7781 e^{-j\omega 2}$$

for $\forall \omega$

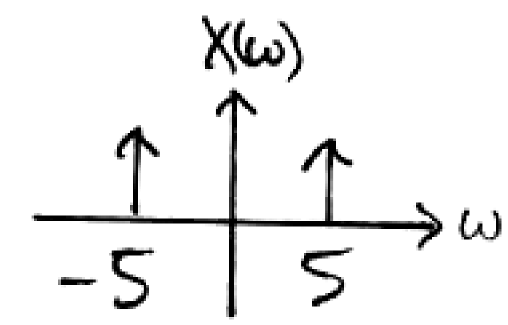




10.1b) $X(j\omega) = \delta(\omega+5) + \delta(\omega-5)$

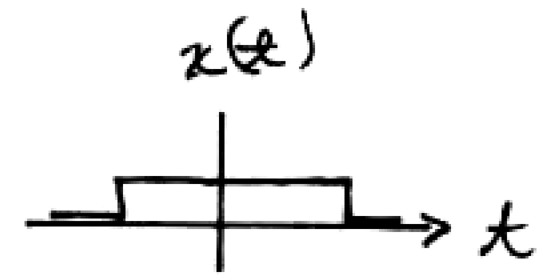
$$x(t) = \mathcal{F}^{-1}\{\delta(\omega+5) + \delta(\omega-5)\} =$$

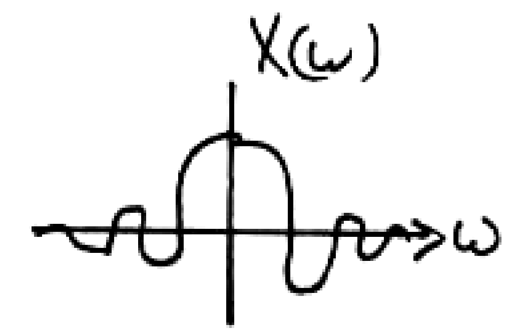
$$= \frac{1}{\pi}\cos(5t)$$



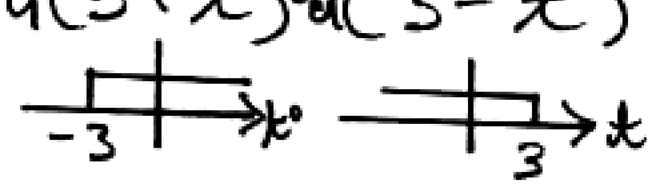
10.1c) $X(j\omega) = 20 \dfrac{\sin(200\omega)}{\omega} = 10 \dfrac{\sin(\omega\, 400/2)}{\omega/2}$

$$x(t) = 10\,u(t+200) - 10\,u(t-200)$$

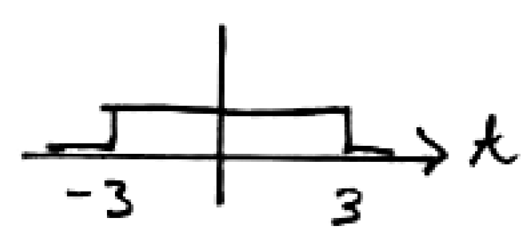


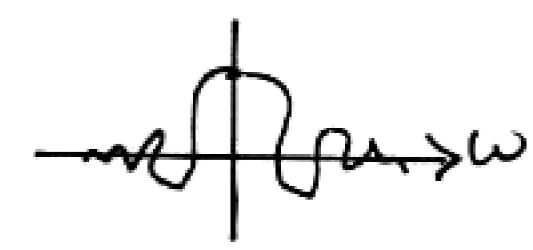


10.1d) $x(t) = u(3+t) \cdot u(3-t) = u(t+3) - u(t-3)$



$$X(j\omega) = \frac{\sin(\omega 3)}{\omega/2}$$





QUIZ 3 CHEAT SHEET SUGGESTIONS:

GRAPHICAL + SYMBOLIC REPRESENTATIONS OF TABLES 12.1 + 12.2 WITH SCALING - COLUMNS IN BOTH TABLES CAN BE SWAPPED (DUALITY) + VARIABLE SWAP

10.2a) $x(t) = \delta(t-5)\cos(t) = \delta(t-5)[\cos(5) \doteq 0.284]$

MULTIPLY

$$\mathcal{F}\{\cos(5)\,\delta(t-5)\} = e^{-j5\omega}\cos(5)\,\mathcal{F}\{\delta(t)\} \doteq e^{-j5\omega}\cdot 0.284$$

10.2b) $x(t) = \delta(t-5) * \cos(t) =$

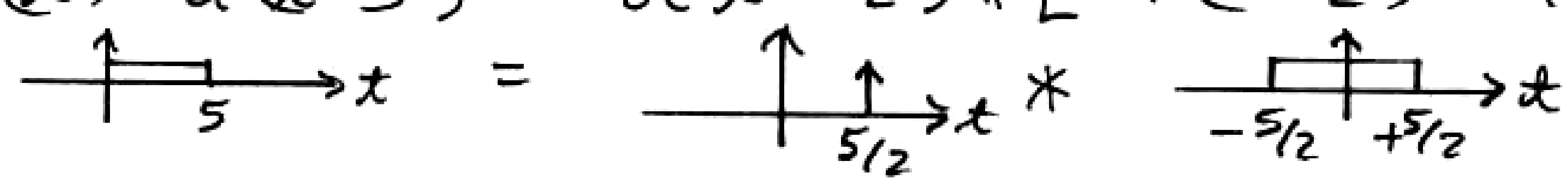
CONVOLVE

$$\mathcal{F}\{\delta(t-5) * \cos(t)\} = \mathcal{F}\{\delta(t-5)\}\cdot\mathcal{F}\{\cos(t)\}$$

FOURIER TRANSFORM PROPERTY

$$= e^{-j5\omega}\cdot 1\cdot[\delta(\omega+1)+\delta(\omega-1)]\cdot\pi$$

10.2c) $x(t) = u(t) - u(t-5) = \delta(t-\frac{5}{2}) * [u(t+\frac{5}{2}) - u(t-\frac{5}{2})]$



$$X(\omega) = \mathcal{F}\{\delta(t-\tfrac{5}{2})\}\cdot\mathcal{F}\{u(t+5/2) - u(t-5/2)\}$$
$$= e^{-j\omega\frac{5}{2}}\cdot\frac{\sin(\omega\, 5/2)}{\omega/2}$$

10.2d) $x(t) = e^{-7t}u(t) - [e^{-t7}u(t-3) = e^{-(t-3)7}u(t-3)e^{-21}]$

$= e^{-7t}[u(t) - u(t-3)]$ ← GOOD TRY BUT $\mathcal{F}\{e^{-7t}\}$ EXPLODES

$= e^{-7t}u(t) - \delta(t-3) * [e^{-(t+3)7}u(t) = e^{-21}e^{-t7}u(t)]$

$$X(j\omega) = \frac{1}{7+j\omega} - \frac{e^{-j\omega 3}\cdot e^{-21}}{7+j\omega} = \frac{1 - e^{-j\omega 3}(e^{-21} \simeq 10^{-9})}{7+j\omega} \simeq \frac{1}{7+j\omega}$$

10.2e) $x(t) = [u(t) - u(t-5)]\cdot\cos(50\pi t)$

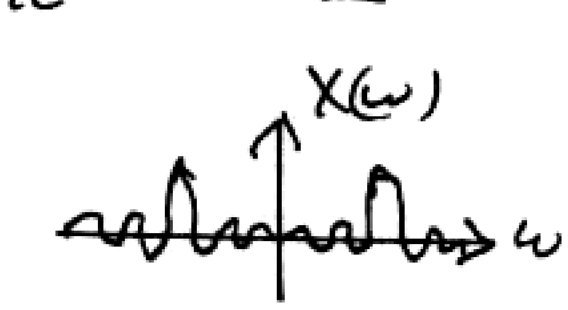
$$= \{\delta(t-\tfrac{5}{2}) * [u(t+\tfrac{5}{2}) - u(t-\tfrac{5}{2})]\}\cdot\cos(50\pi t)$$

$$X(j\omega) = \left[e^{-j\omega\frac{5}{2}}\cdot\frac{\sin(\omega\,5/2)}{\omega/2}\right] * [\delta(\omega+50\pi)+\delta(\omega-50\pi)]\,\pi$$

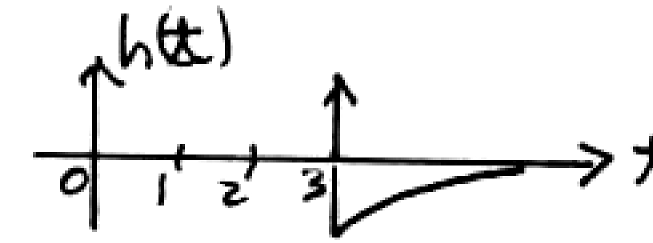
$$= \pi\left[e^{-j\frac{5}{2}(\omega+50\pi)}\frac{\sin[(\omega+50\pi)\frac{5}{2}]}{(\omega+50\pi)/2} + e^{-j\frac{5}{2}(\omega-50\pi)}\frac{\sin[(\omega-50\pi)\frac{5}{2}]}{(\omega-50\pi)/2}\right]$$

$$= \pi e^{j\pi}\left[e^{-j\frac{5\omega}{2}}\frac{\sin[(\omega+50\pi)\frac{5}{2}]}{\frac{\omega}{2}+25\pi} + e^{-j\frac{5\omega}{2}}\frac{\sin[(\omega-50\pi)\frac{5}{2}]}{\frac{\omega}{2}-25\pi}\right]$$

$$= \pi e^{j(\pi-\frac{5\omega}{2})}\left[\frac{\sin(\frac{5}{2}\omega+\pi)}{\frac{\omega}{2}+25\pi} + \frac{(\sin\frac{5}{2}\omega+\pi)}{\frac{\omega}{2}-25\pi}\right]$$



10.3) $h(t) = \delta(t-3) - e^{-7(t-3)} u(t-3) =$



a) $= \delta(t-3) * \left[\delta(t) - e^{-7t} u(t)\right]$

$$H(j\omega) = e^{-j3\omega} \cdot \left[1 - \frac{1}{7+j\omega}\right] = e^{-j\omega 3} \left(\frac{6+j\omega}{7+j\omega}\right)$$

10.3b) SEE MATLAB PLOTS

10.3c) $$x(t) = \underbrace{\underset{\omega=0=DC}{7} + \underset{\omega=7\,[rad/sec]}{7\cos\left(7t + \tfrac{1}{2}\pi\right)}}_{\text{WORK IN FREQUENCY DOMAIN}} + \underbrace{\underset{\text{All frequencies}}{\delta(t-7)}}_{\text{Delay in time} \Leftrightarrow \text{CONV WITH IMPULSE}}$$

$$y(t) = H(0)\cdot 7 + H(7)\cdot 7\cos\left(7t + \tfrac{1}{2}\pi + \angle H(7)\right) + \underbrace{\delta(t-7) * h(t)}$$

$H(0) = \frac{6}{7}$

660.16π

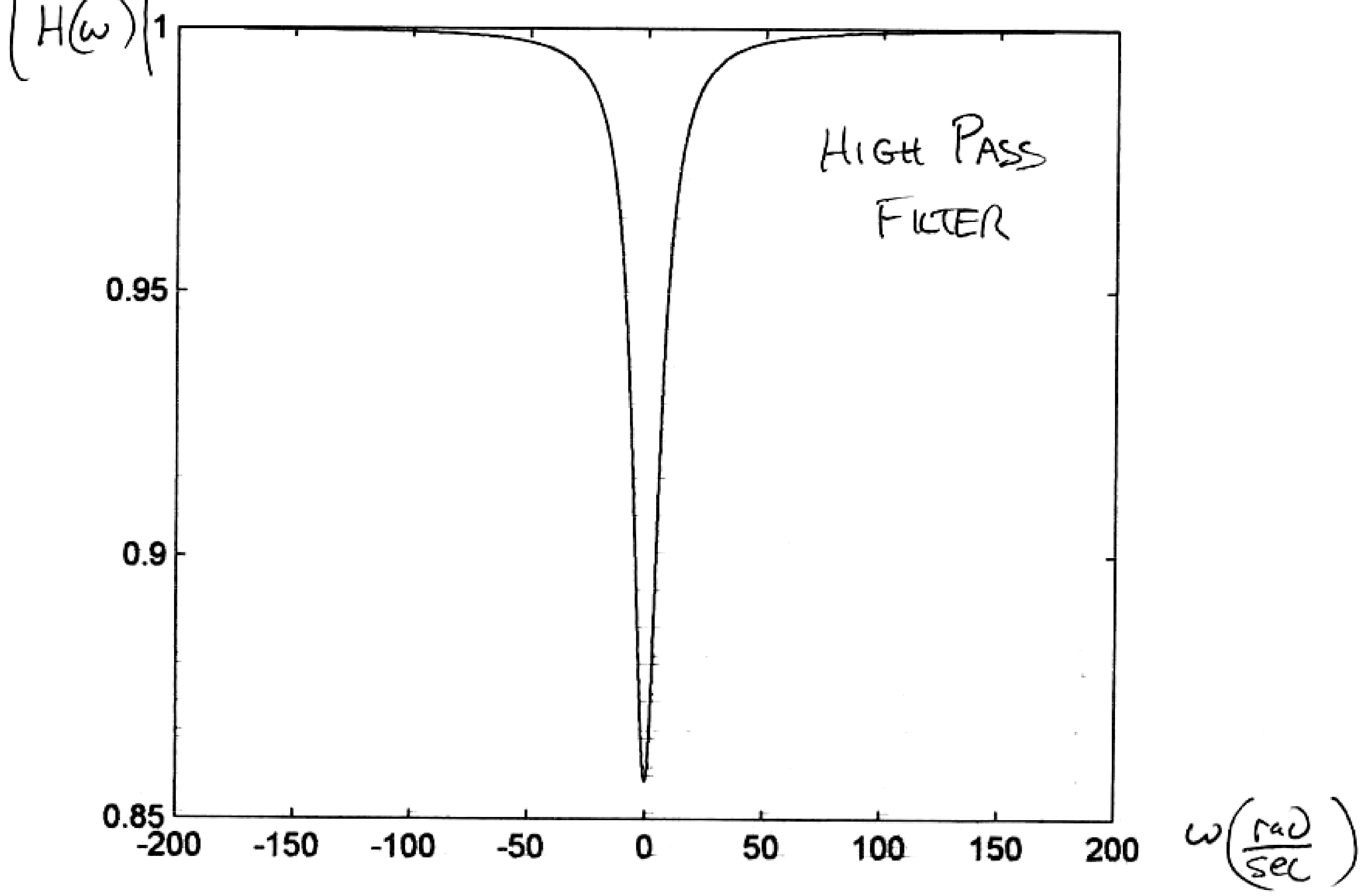
$H(7) \cong 0.9313 \angle\left(-0.6601\pi\ [rad]\right)$

$$\left.\begin{aligned} &\delta(t-7) * \left\{\delta(t-3) - e^{-7(t-3)} u(t-3)\right] \\ &= \delta(t-10) - e^{-7(t-10)} u(t-10) \end{aligned}\right\}$$

$$y(t) = 6 + 6.52\cos(7t - 0.16\pi) + \delta(t-10) - e^{-7(t-10)} u(t-10)$$

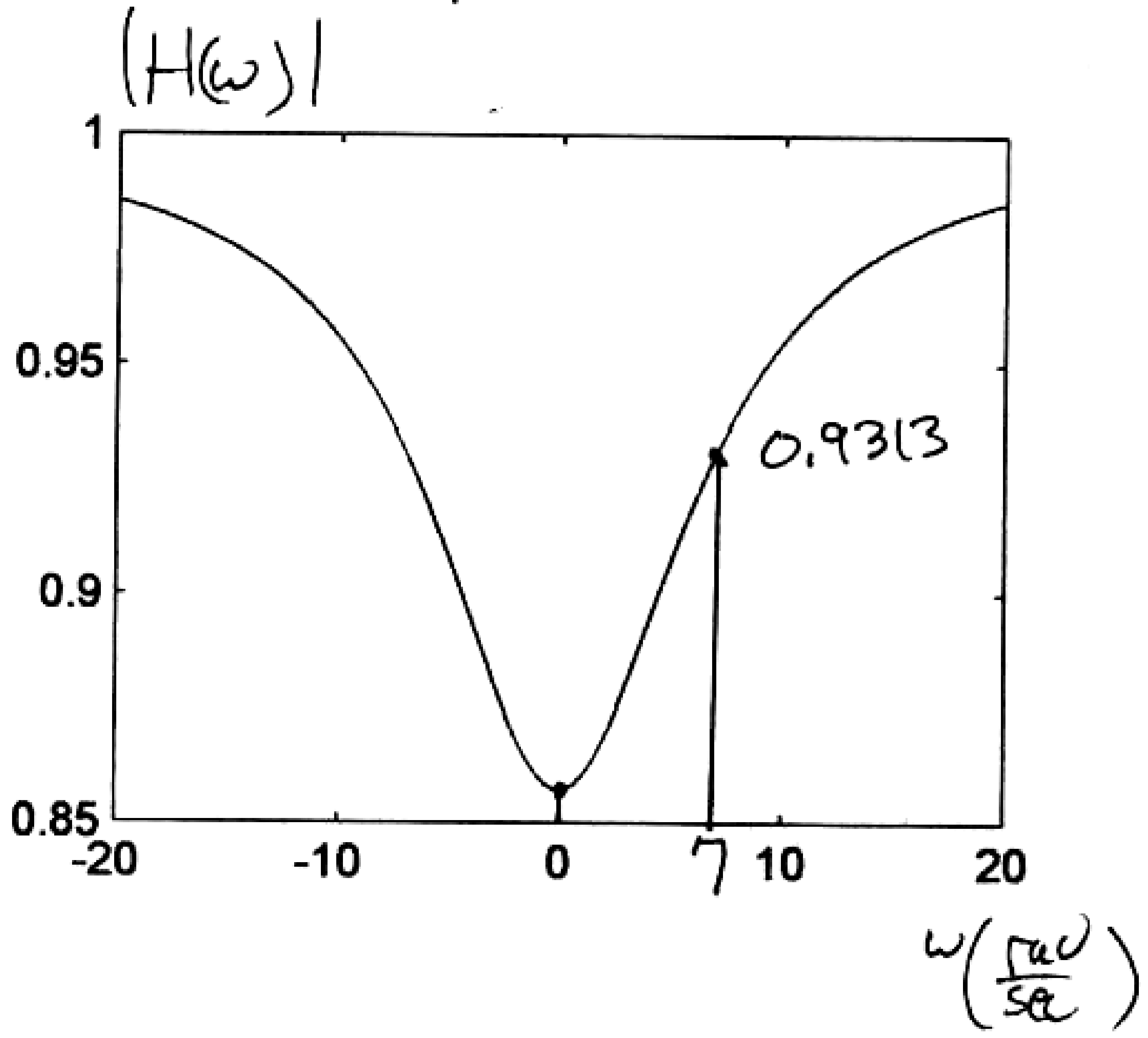
10.3) PLOTS

$$X \longrightarrow \boxed{H(\omega) = \tilde{\chi}} \longrightarrow Y$$

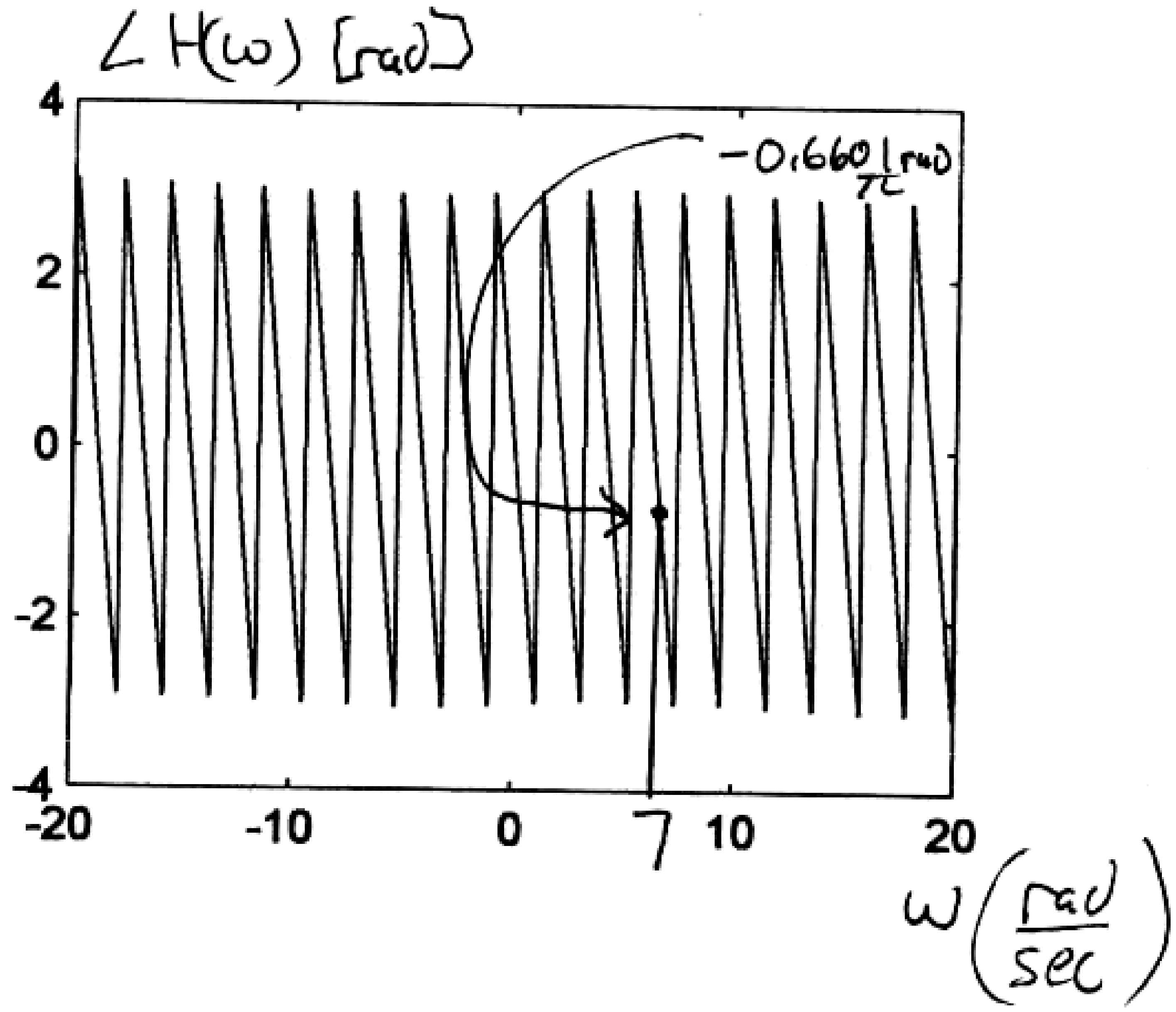


$$H(\omega) = \frac{(6+j\omega)}{(7+j\omega)} \; e^{-j\omega 3}$$

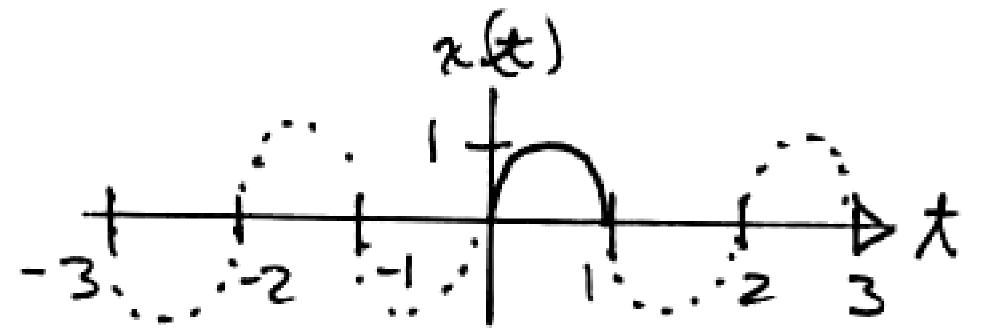
MAGNITUDE



ANGLE



10.4)



a) $X(j\omega) = \int_0^1 \sin(\pi t)\, e^{-j\omega t}\, \partial t$

b)

$$= \frac{1}{2j}\int_0^1 \left(e^{j\pi t} - e^{-j\pi t}\right) e^{-j\omega t}\, \partial t$$

FROM EULER: $\sin(\omega t) = \frac{e^{+j\omega t} - e^{-j\omega t}}{2j}$

$$= \frac{1}{2\cancel{j}}\left[\frac{e^{-jt(\omega-\pi)}}{-\cancel{j}(\omega-\pi)}\Big|_0^1 - \frac{e^{-jt(\omega+\pi)}}{-\cancel{j}(\omega+\pi)}\Big|_0^1\right]$$

$$= \frac{1}{2}\left[\left(\frac{\omega+\pi}{\omega-\pi}\right)\cdot\frac{e^{-j(\omega-\pi)}-1}{(\omega-\pi)} - \frac{(\omega-\pi)}{(\omega-\pi)}\cdot\frac{e^{-j(\omega+\pi)}-1}{(\omega+\pi)}\right]$$

$$= \frac{1}{2(\omega^2-\pi^2)}\Big[\omega e^{-j(\omega-\pi)} - \cancel{\omega} + \pi e^{-j(\omega-\pi)} - \pi - \big(\omega e^{-j(\omega+\pi)} - \cancel{\omega} - \pi e^{-j(\omega+\pi)} + \pi$$

NOTE: $e^{-j(\omega-\pi)} = e^{-j\omega}\left(e^{-j\pi} = -1\right) = -e^{-j\omega}$

AND

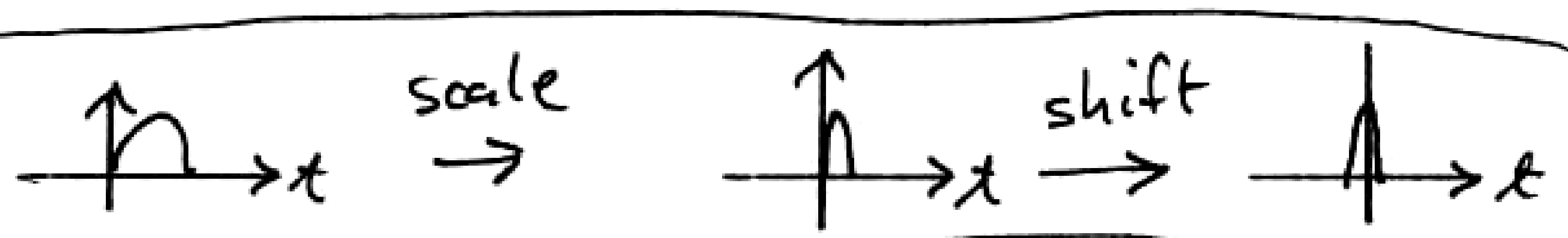
$\cancel{e^{-j}}$ $e^{-j(\omega+\pi)} = e^{-j\omega}\left(e^{j\pi} = -1\right) = -e^{-j\omega}$

$$= -\frac{2\pi}{2}\,\frac{(1+e^{-j\omega})}{(\omega^2-\pi^2)} = \frac{2\pi\,(e^{-j\omega}+1)}{\pi^2-\omega^2} = \frac{2\pi}{(\pi^2-\omega^2)2}\, e^{-j\frac{\omega}{2}}\left(e^{-j\frac{\omega}{2}} + e^{j\frac{\omega}{2}}\right)$$

$$= \frac{2\pi\cos(\omega/2)}{\pi^2-\omega^2}\cdot e^{-j\frac{\omega}{2}} = |X(\omega)|\cdot\angle X(\omega)$$

c) SEE MATLAB PLOTS

d) SCALE THEN SHIFT:



SCALED:

$$\sin(100t)\Big|_0^{0.01} \leftrightarrow \frac{1}{100}\,\frac{2\pi\cos\left(\frac{\omega}{200}\right)}{\pi^2-\left(\frac{\omega}{100}\right)^2}\, e^{-j\frac{\omega}{200}}$$

USE PROPERTIES:

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

$$x(t-t_0) \leftrightarrow e^{-j\omega t_0}X(\omega)$$

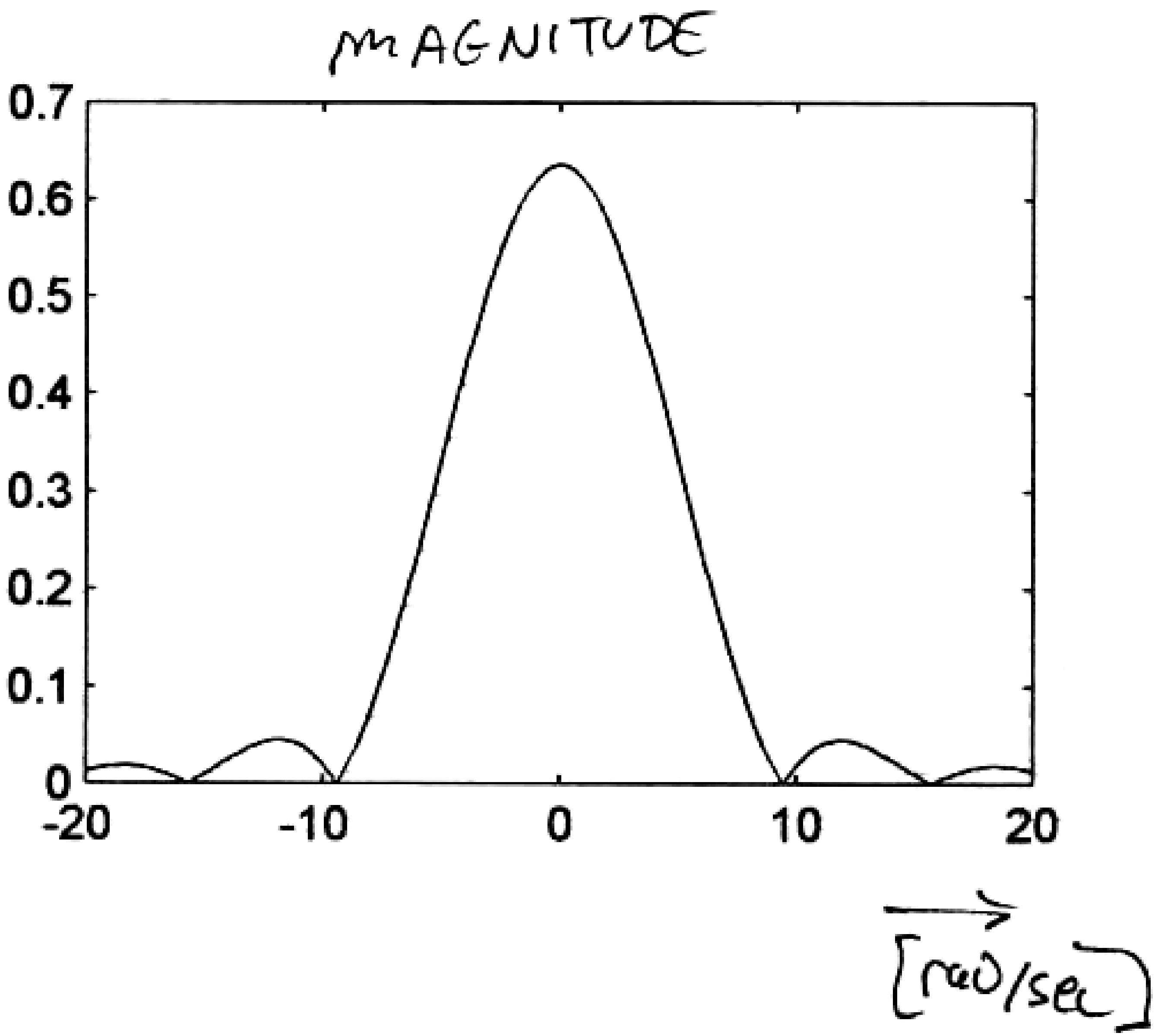
SHIFTED:

$$\cos(100t)\Big|_{-0.005}^{0.005} \leftrightarrow \frac{e^{j\omega\frac{1}{200}}}{100}\,\frac{2\pi\cos\left(\frac{\omega}{200}\right)}{\pi^2-\left(\frac{\omega}{100}\right)^2}\, e^{-j\frac{\omega}{200}} = \frac{1}{100}\,\frac{2\pi\cos\left(\frac{\omega}{200}\right)}{\pi^2-\left(\frac{\omega}{100}\right)^2}$$

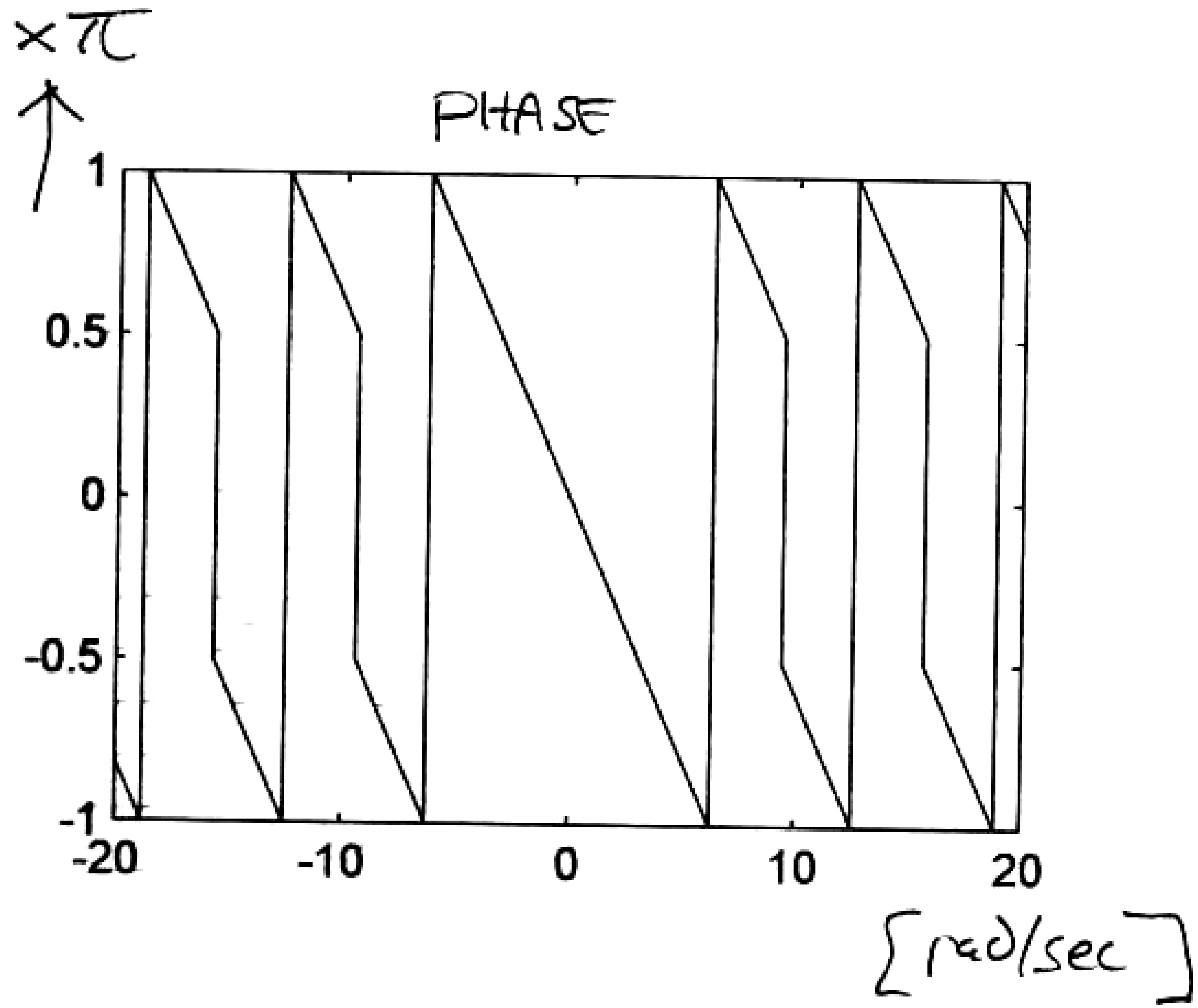
e) NOTE THAT EXP() TERMS CANCEL SO THAT:

$$\cos(100t)\left[u\left(t+\tfrac{1}{200}\right)-u\left(t-\tfrac{1}{200}\right)\right] \leftrightarrow \frac{1}{100}\,\frac{2\pi\cos\left(\frac{\omega}{200}\right)}{\pi^2-\left(\frac{\omega}{100}\right)^2} = \text{All Real}$$

10.4 PLOTS



MAGNITUDE
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-20
-10
0
10
20
[rad/sec]



×π
PHASE
1
0.5
0
-0.5
-1
-20
-10
0
10
20
[rad/sec]

Prob 10.5

(a) $$a_k = \frac{1}{T_0}\int_{-0.1}^{0.1} 100 e^{-j2\pi k t/T_0}\,dt \qquad T_0 = 0.4 = \tfrac{2}{5}\text{ sec}$$

$$\boxed{\frac{2\pi}{T_0} = 5\pi \text{ rad/s}}$$

$$= \frac{5}{2}(100)\left.\frac{e^{-j5\pi k t}}{-j5\pi k}\right|_{-1/10}^{1/10}$$

$$a_k = 50\left(\frac{e^{-j\pi k/2} - e^{j\pi k/2}}{-j\pi k}\right) = 100\frac{\sin(\pi k/2)}{\pi k}$$

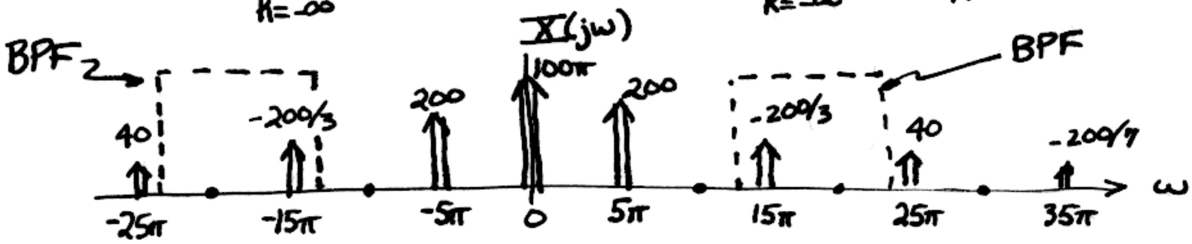
Notice the "sinc" formula

Note: $a_0 = \lim_{k\to 0} a_k = 100\cdot\frac{1}{2} = 50$

(b) Fundamental Freq $= 5\pi$ rad/s

So we have impulses at $\pm 5\pi$, $\pm 10\pi$, $\pm 15\pi$, $\pm 20\pi$ rad/sec.

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k\,\delta(\omega - 5\pi k) = \sum_{k=-\infty}^{\infty} \frac{200\sin(\pi k/2)}{k}\,\delta(\omega - 5\pi k)$$



(c) Only the impulses at $\omega = \pm 15\pi$ rad/s will pass thru the filter.

$$Y(j\omega) = -\frac{200}{3}\delta(\omega - 15\pi) - \frac{200}{3}\delta(\omega + 15\pi)$$

If $y(t)$ has a Fourier Series with coeffs $\{b_k\}$, then it only has $b_{\pm 3}$. and $b_3 = -\frac{200}{6\pi} = -\frac{100}{3\pi}$; $b_{-3} = -100/3\pi$.

(d) $$y(t) = -\frac{100}{3\pi}e^{j15\pi t} - \frac{100}{3\pi}e^{-j15\pi t} = -\frac{200}{3\pi}\cos(15\pi t) = \frac{200}{3\pi}\cos(15\pi t + \pi)$$

(e) This High-pass filter removes DC.

$y(t) = x(t) - a_0$ because $Y(j\omega) = X(j\omega) - 2\pi a_0\,\delta(\omega)$