GEORGIA INSTITUTE OF TECHNOLOGY

SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

**ECE 2025 Fall 1999**

**Problem Set #12**

Assigned: 12 November

Due Date: 19 November 99 (FRIDAY)

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## PROBLEM 12.1*:

Try your hand at expressing each of the following in a simpler form:

(a) Use the fact that $h(t)\delta(t-t_0) = h(t_0)\delta(t-t_0)$ to derive

$$\begin{aligned}
[e^{-t}\cos(2\pi t)][\delta(t)+2\delta(t-3)] &= [e^{-t}\cos(2\pi t)\delta(t)] + [e^{-t}\cos(2\pi t)2\delta(t-3)] \\
&= [e^{-0}\cos(2\pi\cdot 0)\delta(t)] + [e^{-3}\cos(2\pi\cdot 3)2\delta(t-3)] \\
&= [1\cdot 1\cdot\delta(t)] + [e^{-3}\cdot 1\cdot 2\delta(t-3)] \\
&= \delta(t) + 2e^{-3}\delta(t-3)
\end{aligned}$$

(b) Use the fact that $h(t) * \delta(t-t_0) = h(t_0)$ to derive

$$\begin{aligned}
& [\delta(t) - u(t)] * [\delta(t-1) + \delta(t-2)] \\
= \;& [\delta(t) * \delta(t-1)] + [\delta(t) * \delta(t-2)] - [u(t) * \delta(t-1)] - [u(t) * \delta(t-2)] \\
= \;& \delta(t-1) + \delta(t-2) - u(t-1) - u(t-2)
\end{aligned}$$

(c) Use the fact that $\int_{-\infty}^{\infty} h(\tau)\delta(t-\tau)d\tau = h(t)$ to derive

$$\int_{-\infty}^{\infty} \sin(2\pi\tau + \pi/3)\delta(t-\tau)d\tau = \sin(2\pi t + \pi/3)$$

(d) Take the derivative of the product and use the fact that $h(t)\delta(t-t_0) = h(t_0)\delta(t-t_0)$ to derive

$$\begin{aligned}
\frac{d}{dt}\left[e^{-(t-1)}u(t-1)\right] &= e^{-(t-1)}\frac{du(t-1)}{dt} + \frac{de^{-(t-1)}}{dt}u(t-1) \\
&= e^{-(t-1)}\delta(t-1) - e^{-(t-1)}u(t-1) \\
&= e^{0}\delta(t-1) - e^{-(t-1)}u(t-1) \\
&= \delta(t-1) - e^{-(t-1)}u(t-1)
\end{aligned}$$

## PROBLEM 12.2*:

A linear time-invariant system has impulse response

$$h(t) = \delta(t) - 10e^{-10t}u(t)$$

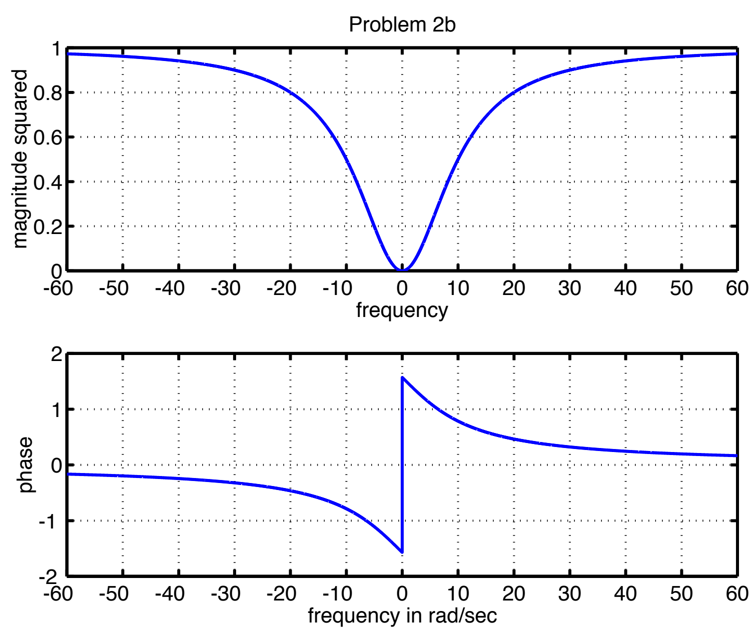
(a) Determine the frequency response $H(j\omega)$ of the system. Express your answer as a rational function of $(j\omega)$.

$$\begin{aligned}
H(j\omega) &= \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt - \int_{-\infty}^{\infty} 10e^{-10t}u(t)e^{-j\omega t}dt \\
&= e^{-j\omega\cdot 0} - 10\int_{0}^{\infty} e^{-(10+j\omega)t}dt \\
&= 1 - \left.\frac{10e^{-(10+j\omega)t}}{-(10+j\omega)}\right|_0^{\infty} \\
&= 1 - \left[0 - \frac{10}{-(10+j\omega)}\right] = 1 - \frac{10}{10+j\omega} = \frac{j\omega}{10+j\omega}
\end{aligned}$$

(b) Plot $|H(j\omega)|^2 = H(j\omega)H^*(j\omega)$ and $\angle H(j\omega)$ as a function of $\omega$.

$$\begin{aligned}
|H(j\omega)|^2 &= \frac{j\omega}{10+j\omega}\cdot\frac{-j\omega}{10-j\omega} \\
&= \frac{\omega^2}{100+\omega^2}
\end{aligned}$$

$$\begin{aligned}
\angle H(j\omega) &= \angle\left(\frac{j\omega}{10+j\omega}\cdot\frac{10-j\omega}{10-j\omega}\right) \\
&= \angle\left(\frac{\omega^2+j10\omega}{100+\omega^2}\right) = \tan^{-1}\left(\frac{10\omega}{\omega^2}\right) = \tan^{-1}\left(\frac{10}{\omega}\right)
\end{aligned}$$



(c) Use superposition to find the output due to the input

$$\begin{aligned} x(t) &= 100 + 20\cos(10t) + \delta(t-3) \\ &= 100 + 10e^{j10t} + 10e^{-j10t} + \delta(t-3) \end{aligned}$$

First compute the frequency response for each of the frequencies (0, 10, and −10 rad/sec):

$$\begin{aligned} H(j0) &= \frac{j0}{10+j0} = 0 \\ H(j10) &= \frac{j10}{10+j10} = \frac{1+j}{2} = \frac{\sqrt{2}}{2}e^{j\pi/4} \\ H(-j10) &= H^*(j10) = \frac{1-j}{2} = \frac{\sqrt{2}}{2}e^{-j\pi/4} \end{aligned}$$

Then use frequency response and impulse response to derive

$$\begin{aligned} y(t) &= H(j0)100 + H(j10)10e^{j10t} + H(-j10)10e^{-j10t} + h(t) * \delta(t-3) \\ &= \frac{\sqrt{2}}{2}e^{j\pi/4}10e^{j10t} + \frac{\sqrt{2}}{2}e^{-j\pi/4}10e^{-j10t} + h(t-3) \\ &= 5\sqrt{2}e^{j(10t+\pi/4)} + 5\sqrt{2}e^{-j(10t+\pi/4)} + \delta(t-3) - 10e^{-10(t-3)}u(t-3) \\ &= 10\sqrt{2}\cos(10t+\pi/4) + \delta(t-3) - 10e^{-10(t-3)}u(t-3) \end{aligned}$$

**PROBLEM 12.3*:**

An LTI system is defined by the following input/output relation:

$$y(t) = 0.5x(t) + x(t-4) + 0.5x(t-8) \tag{1}$$

(a) Determine the impulse response $h(t)$ of the overall system; i.e., determine the output when the input is an impulse.

$$h(t) = 0.5\delta(t) + \delta(t-4) + 0.5\delta(t-8)$$

(b) Substitute your answer for $h(t)$ into the the integral formula

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

to obtain the frequency response. Use the impulse property: $\int_{-\infty}^{\infty} \delta(t-t_d)f(t)dt = f(t_d)$.

$$\begin{aligned}
H(j\omega) &= \int_{-\infty}^{\infty} 0.5\delta(t)e^{-j\omega t}dt + \int_{-\infty}^{\infty} \delta(t-4)e^{-j\omega t}dt + \int_{-\infty}^{\infty} 0.5\delta(t-8)e^{-j\omega t}dt \\
&= 0.5 + e^{-j4\omega} + 0.5e^{-j8\omega} \\
&= 0.5e^{-j4\omega}\left(e^{j4\omega} + 2 + e^{-j4\omega}\right) \\
&= 0.5e^{-j4\omega}\left(e^{j2\omega} + e^{-j2\omega}\right)^2 \\
&= \cos^2(2\omega)e^{-j4\omega}
\end{aligned}$$

(c) Apply the system definition given in Eq. (1) directly to the input $x(t) = e^{j\omega t}$ for $-\infty < t < \infty$ and show that $y(t) = H(j\omega)e^{j\omega t}$, where $H(j\omega)$ is as determined in part (b).

If we input $x(t) = e^{j\omega t}$ into Eq. (1), the result is

$$\begin{aligned}
y(t) &= 0.5x(t) + x(t-4) + 0.5x(t-8) \\
&= 0.5e^{j\omega t} + e^{j\omega(t-4)} + 0.5e^{j\omega(t-8)} \\
&= (0.5 + e^{-j4\omega} + 0.5e^{-j8\omega})e^{j\omega t}
\end{aligned}$$

On the other hand, if we apply the frequency response to $x(t) = e^{j\omega t}$, the result is

$$\begin{aligned}
y(t) &= H(j\omega)e^{j\omega t} \\
&= (0.5 + e^{-j4\omega} + 0.5e^{-j8\omega})e^{j\omega t} \\
&= 0.5e^{j\omega t} + e^{j\omega(t-4)} + 0.5e^{j\omega(t-8)}
\end{aligned}$$

As expected, the two methods give the same result.

(d) Sketch the magnitude and phase $|H(j\omega)|$ and $\angle H(j\omega)$ as functions of $\omega$.

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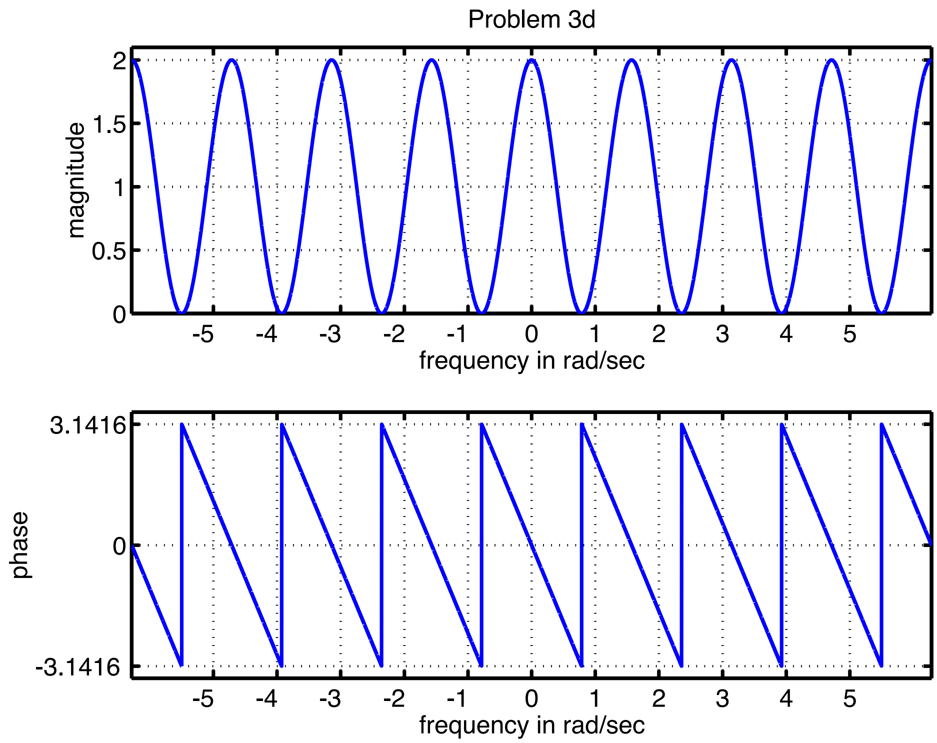
From the frequency response $H(j\omega) = \cos^2(2\omega)e^{-j4\omega}$, it is clear that $|H(j\omega)| = \cos^2(2\omega)$ and $\angle H(j\omega) = -4\omega$.

Notice that the frequency response is equal to zero at $\omega = \pm\pi/4, \pm 3\pi/4, \pm 5\pi/4$, etc.

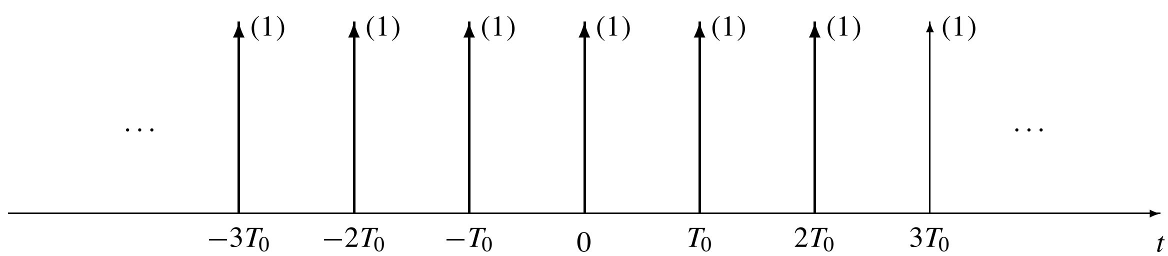


**PROBLEM 12.4:**

A *periodic impulse train* with period $T_0$ is defined to be the signal

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0).$$

(a) Plot this signal for $-3T_0 \le t \le 3T_0$.




(b) What is the fundamental frequency, $\omega_0$ if $T_0 = 10$?

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{10} = \frac{\pi}{5}$$

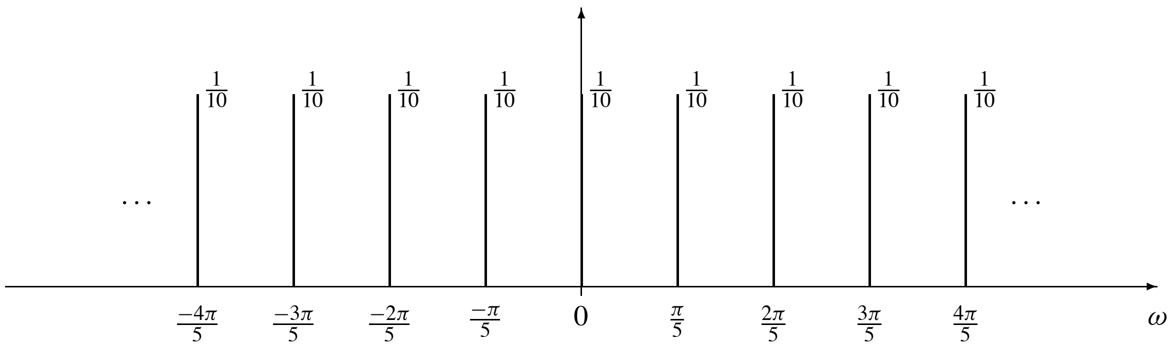
(c) Determine the Fourier coefficients $a_k$ in the Fourier series representatation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

By integrating over the period centered at zero, we get the following equation:

$$\begin{aligned}
a_k &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jk\omega_0 t}dt \\
&= \frac{1}{10}\int_{-5}^{5} \delta(t)e^{-jk(\pi/5)t}dt \\
&= \frac{1}{10}\int_{-5}^{5} \delta(t)e^{-jk(\pi/5)(0)}dt \\
&= \frac{1}{10}\int_{-5}^{5} \delta(t)dt \\
&= \frac{1}{10}
\end{aligned}$$

(d) Plot the spectrum of this signal for $-4\omega_0 \le \omega \le 4\omega_0$.




(e) The periodic impulse train $x(t)$ is the input to a system with frequency response

$$H(j\omega) = \begin{cases} e^{-j\omega 4} & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co}. \end{cases}$$

Determine the output signal $y(t)$ if $\omega_{co} = \pi/T_0$.

This is a "delay-by-4" system. If $\omega_{co} = \pi/10$, the only nonzero output frequency component is that at $\omega = 0$ (the $k = 0$ term). This is the DC term only. Thus, the output is

$$y(t) = H(j0)0.1e^{-j(0)(\pi/5)t} = 0.1$$

and the output signal is DC.

(f) The periodic impulse train $x(t)$ is the input to a system with frequency response

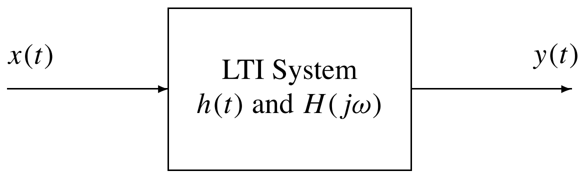
$$H(j\omega) = \begin{cases} e^{-j\omega 4} & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co}. \end{cases}$$

Determine the output signal $y(t)$ if $\omega_{co} = 3\pi/T_0$.

This is a "delay-by-4" system. In this case, with $\omega_{co} = 3\pi/10$, the terms at $k = 0, 1, -1$ are all included in the output. Thus, the output is

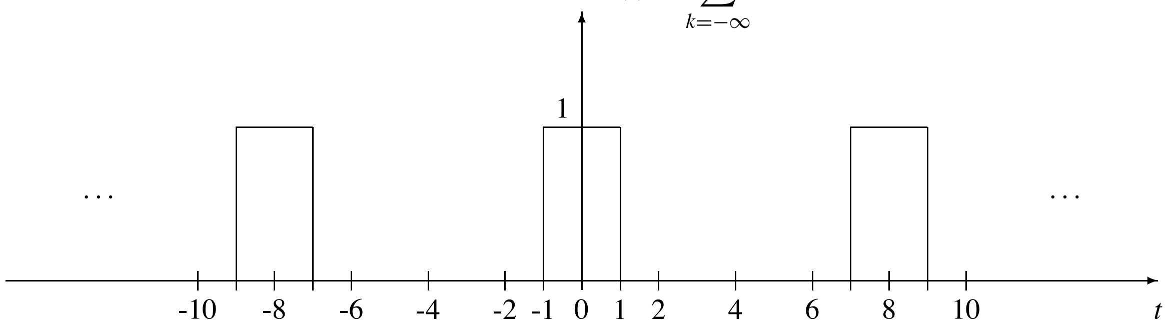
$$\begin{aligned} y(t) &= H(j0)0.1e^{-j(0)(\pi/5)t} + H(j\pi/5)0.1e^{j(1)(\pi/5)t} + H(-j\pi/5)0.1e^{j(-1)(\pi/5)t} \\ &= 0.1 + 0.1e^{-j4\pi/5}e^{j(\pi/5)t} + 0.1e^{j4\pi/5}e^{-j(\pi/5)t} \\ &= 0.1\left[1 + 2\cos\left(\frac{\pi}{5}t - \frac{4\pi}{5}\right)\right] \\ &= 0.1\left[1 + 2\cos\left(\frac{\pi}{5}(t-4)\right)\right] \end{aligned}$$

**PROBLEM 12.5*:**
Consider the LTI system below:




The input to this system is the periodic pulse wave $x(t)$ depicted below:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$



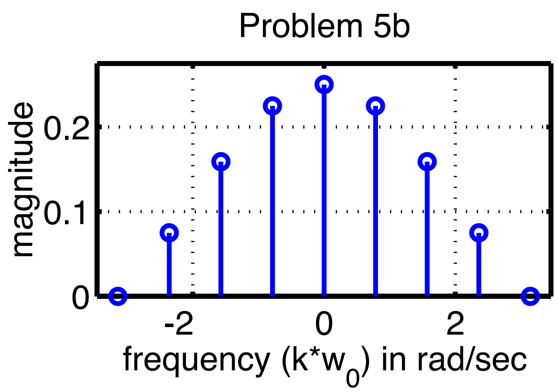

(a) Determine $\omega_0$ and the coefficients $a_k$ in the Fourier series representation of $x(t)$. First, find $T_0$ and $\omega_0$ from the graph:

$$T_0 = 8 \qquad \omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{4}$$

Next, compute the Fourier coefficients by preforming the coefficient integral over one period of the waveform. *Note: In this case we chose a period that includes the pulse centered at 0, and thus we only need integrate over the pulse because the signal value is equal to 0 otherwise.*

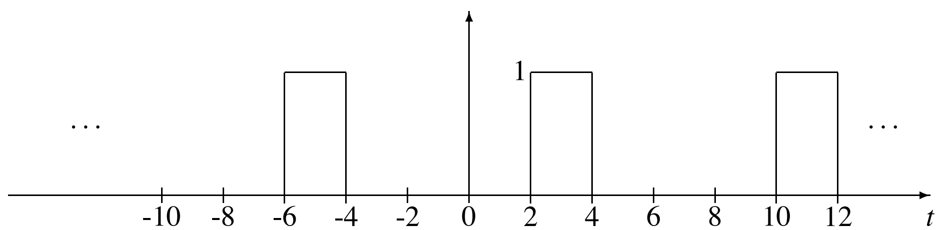
$$\begin{aligned} a_k &= \frac{1}{T_0}\int_{-1}^{1}(1)e^{-jk\omega_0 t}dt \\ &= \frac{1}{8}\int_{-1}^{1} e^{-jk(\pi/4)t}dt \\ &= \frac{1}{8}\left[\frac{e^{-jk(\pi/4)t}}{-jk\pi/4}\right]_{-1}^{1} \\ &= \frac{e^{-jk\pi/4} - e^{jk\pi/4}}{-2jk\pi} \\ &= \frac{\sin(k\pi/4)}{k\pi} \end{aligned}$$

(b) Plot the spectrum of the signal $x(t)$; i.e., make a plot showing the $a_k$s plotted at the frequencies $k\omega_0$ for $-4\omega_0 \leq \omega \leq 4\omega_0$.




(c) If the frequency response is $H(j\omega) = e^{-j\omega 3}$, plot the output of the system $y(t)$ when the input is $x(t)$ as plotted above.

We can solve this problem by realizing that $H(j\omega) = e^{-j\omega 3}$ is the frequency response for $h(t) = \delta(t-3)$. Thus, $y(t) = x(t) * \delta(t-3) = x(t-3)$. This is a "delay-by-3" system.




(d) If the frequency response of the system is the ideal lowpass filter

$$H(j\omega) = \begin{cases} 1 & |\omega| < 0.5\pi \\ 0 & |\omega| > 0.5\pi \end{cases}$$

what is the output of the system when the input is $x(t)$ as depicted above? Give an equation for $y(t)$.

*Since the frequency response is not defined at $\omega = \frac{1}{2}\pi$, there are two acceptable answers for this part.*

Assuming that $H(j\omega) = 0$ when $\omega = \pi/2$, the only frequency components that are present are the DC term and the adjacent terms (at $\omega = \pm\pi/4$), or, in other words the terms with $|k| \leq 1$. Thus the output is just the sum of these components:

$$\begin{aligned} y(t) &= 0.25 + \frac{\sqrt{2}}{2\pi} e^{j(\pi/4)t} + \frac{\sqrt{2}}{2\pi} e^{-j(\pi/4)t} \\ &= 0.25 + \frac{\sqrt{2}}{\pi} \cos\left(\frac{\pi}{4} t\right) \end{aligned}$$

Assuming that $H(j\omega) = 1$ when $\omega = \pi/2$, then five frequency components are present: the DC term and the adjacent terms at $\omega = \pm\pi/4$ and also at $\omega = \pm\pi/2$, or, in other words the terms with $|k| \leq 2$. Thus the output is just the sum of these components:

$$\begin{aligned} y(t) &= 0.25 + \frac{\sqrt{2}}{2\pi} e^{j(\pi/4)t} + \frac{\sqrt{2}}{2\pi} e^{-j(\pi/4)t} + \frac{1}{2\pi} e^{j(\pi/2)t} + \frac{1}{2\pi} e^{-j(\pi/2)t} \\ &= 0.25 + \frac{\sqrt{2}}{\pi} \cos\left(\frac{\pi}{4} t\right) + \frac{1}{\pi} \cos\left(\frac{\pi}{2} t\right) \end{aligned}$$

## PROBLEM 12.6*:

Use the delay property of Fourier transforms,

$$x(t-t_d) \Longleftrightarrow e^{-j\omega t_d} X(j\omega),$$

to determine the Fourier transforms of the following signals:

(a) $x(t) = \delta(t-5)$

Given the fact that the Fourier transform of $\delta(t)$ is equal to 1, the Fourier transform of this delayed impulse is $X(j\omega) = e^{-j5\omega}$.

(b) $x(t) = 20\dfrac{\sin(200\pi(t-10))}{\pi(t-10)}$

The Fourier transform of $\sin(200\pi t)/\pi t$ is equal to the box defined by difference of step functions: $u(\omega+200\pi) - u(\omega-200\pi)$. $x(t)$ is equal to this function scaled (in amplitude) by a factor of 20 and delayed by 10 seconds. Thus, the Fourier transform of $x(t)$ is

$$X(j\omega) = 20e^{-j10\omega}[u(\omega+200\pi) - u(\omega-200\pi)]$$

(c) $x(t) = e^{-4t}u(t) - e^{-4t}u(t-10) = e^{-4t}u(t) - e^{-40}e^{-4(t-10)}u(t-10)$

The Fourier transform of $e^{-4t}u(t)$ is $1/(4+j\omega)$. Thus, the Fourier transform of $x(t)$ is

$$\begin{aligned} X(j\omega) &= \frac{1}{4+j\omega} - e^{-40}e^{-j10\omega}\frac{1}{4+j\omega} \\ &= \frac{1-e^{-j10\omega-40}}{4+j\omega} \end{aligned}$$