GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 2025 Fall 1999
Problem Set #9

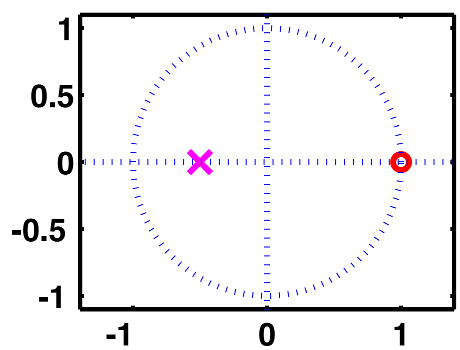
Assigned: 22 Oct 99
Due Date: 29 Oct 1999 (FRIDAY)

PROBLEM 9.1:

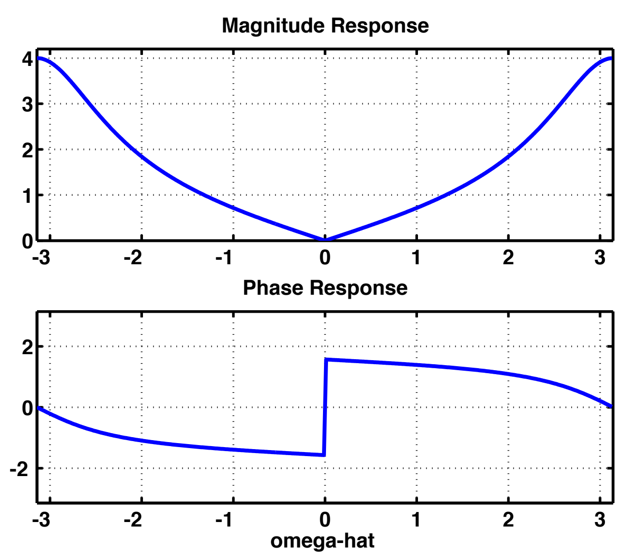
(a) First of all, we put the two terms over a common denominator: $H(z) = \dfrac{1}{(1+0.5z^{-1})} - \dfrac{z^{-1}}{(1+0.5z^{-1})} = \dfrac{(1-z^{-1})}{(1+0.5z^{-1})}$.
Since $H(z) = Y(z)/X(z)$, we can write $Y(z)(1+0.5z^{-1}) = X(z)(1-z^{-1})$ which becomes

$$y[n] + 0.5y[n-1] = x[n] - x[n-1]$$

(b) We have one pole at $z = -0.5$, and one zero at $z = 1$ because $H(z) = \dfrac{1-z^{-1}}{1+0.5z^{-1}} = \dfrac{z-1}{z-(-0.5)}$. The pole-zero plot in the z-domain is below:



(c) Evaluate $H(z)$ at $z = e^{j\omega}$ and we get $H(e^{j\hat{\omega}}) = \dfrac{1-e^{-j\hat{\omega}}}{1+0.5e^{-j\hat{\omega}}} = \dfrac{2j\sin(\hat{\omega}/2)e^{-j\hat{\omega}/2}}{1+0.5e^{-j\hat{\omega}}}$
The plot of this frequency response:



Notice that we have a phase shift by π at $\hat{\omega} = 0$ because $\sin(\hat{\omega}/2)$ in the numerator changes sign at this location. Use the MATLAB function `freqz()` to generate this plot.

```
om = -pi:pi/100:pi;
H1 = freqz([1,-1],[1,0.5],om);
subplot(2,1,1),plot(om,abs(H1)); axis tight
subplot(2,1,2),plot(om,angle(H1));
axis([-pi pi -pi pi]);
```

(d) Use the geometric sequence sum to do the z-transform inversion:

$$\frac{1}{1+0.5z^{-1}} = \sum_{k=0}^{\infty}(-0.5z^{-1})^k = \sum_{k=0}^{\infty}(-0.5)^k z^{-k}$$

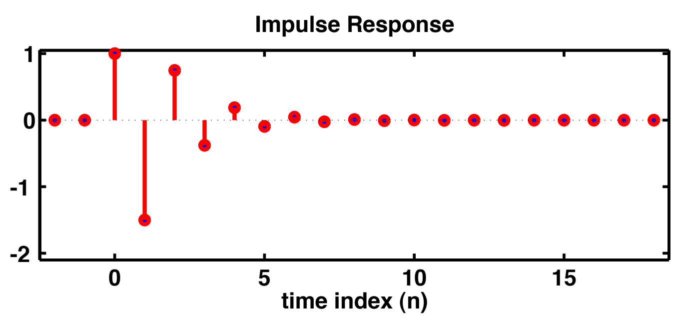
Therefore,

$$H(z) = \frac{1}{1+0.5z^{-1}} - \frac{z^{-1}}{1+0.5z^{-1}} = \sum_{k=0}^{\infty}(-0.5)^k z^{-k} - z^{-1}\sum_{k=0}^{\infty}(-0.5)^k z^{-k}$$

As a result of the shifting property, the impulse response is $h[n] = (-0.5)^n\, u[n] - (-0.5)^{n-1}\, u[n-1]$, or

$$h[n] = \begin{cases} h[n] = 0 & n < 0 \\ h[n] = 1 & n = 0 \\ h[n] = (-0.5)^n - (-0.5)^{n-1} & n > 0 \end{cases}$$

Plot of this impulse response:



(e) We have three components. The first component is a DC (zero frequency) term, and therefore its response = 0, because $H(e^{j\hat{\omega}})$ is zero at $\hat{\omega} = 0$. The second component is the impulse response, which we solved above, multiplied by 7. The third component is a term at $\hat{\omega} = \pi/2$, and therefore the resulting gain is 1.265 and phase shift of 0.4π. The resulting expression for $y[n]$ is

$$y[n] = 7(-0.5)^n\, u[n] - 7(0.5)^{n-1}\, u[n-1] + 12.65\cos(0.5\pi n + 0.4\pi)$$

or, equivalently

$$y[n] = 7\delta[n] - 10.5(0.5)^{n-1}\, u[n-1] + 12.65\cos(0.5\pi n + 0.4\pi)$$

or, equivalently

$$y[n] = \begin{cases} y[n] = 12.65\cos(0.5\pi n + 0.4\pi) & n < 0 \\ y[n] = 7 + 12.65\cos(0.5\pi n + 0.4\pi) & n = 0 \\ y[n] = 7\left((-0.5)^n - (-0.5)^{n-1}\right) + 12.65\cos(0.5\pi n + 0.4\pi) & n > 0 \end{cases}$$

PROBLEM 9.2:

(a) If we want the output to be equal to the input, then for the z-transform we require $Y(z) = X(z)$. Therefore, the combined transfer function must be equal to one, $H(z) = H_1(z)H_2(z) = 1$; therefore,

$$H_2(z) = 1/H_1(z).$$

(b) If $H_1(z)$ is an FIR filter, then $H_2(z)$ is an IIR filter, because the zeros of $H_1(z)$ become the poles of $H_2(z)$,In other words, the numerator of $H_1(z)$ becomes the denominator of $H_2(z)$.

(c) First, recall summation formula for a geometric sequence: $\displaystyle\sum_{k=0}^{N} r^k = \frac{1 - r^{N+1}}{1 - r}$. Since the recursive difference equation for $w[n]$ is given by

$$w[n] = \sum_{k=0}^{9} \alpha^k x[n-k],$$

we can take the z-transform of the difference equation (using the shifting property) to get the system function of the first filter:

$$W(z) = \sum_{k=0}^{9} \alpha^k z^{-k} X(z)$$

$$H_1(z) = \frac{W(z)}{X(z)} = \sum_{k=0}^{9} (\alpha z^{-1})^k = \frac{1 - \alpha^{10} z^{-10}}{1 - \alpha z^{-1}}$$

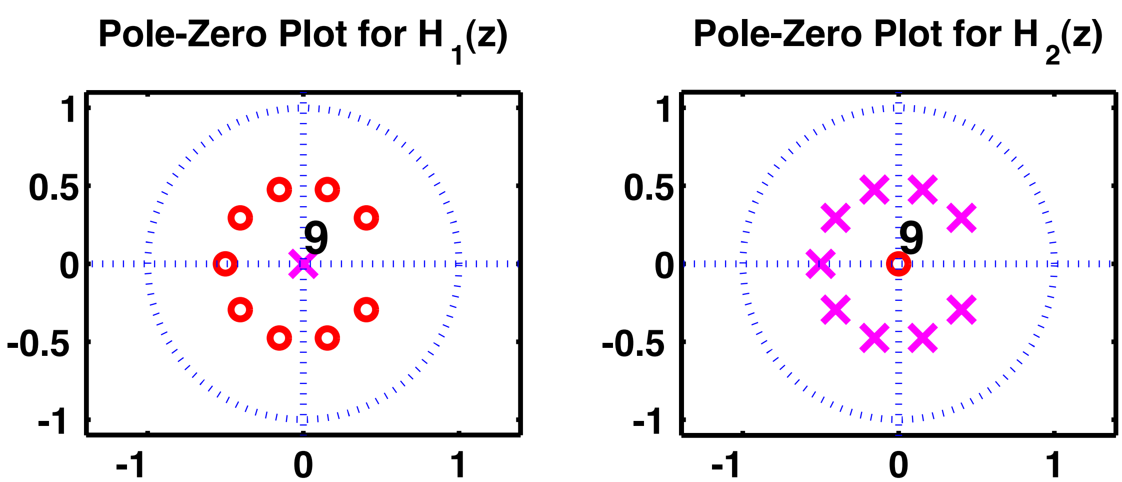
For this system function, we can write $H_1(z)$ with positive powers of z

$$H_1(z) = \frac{z^{10} - \alpha^{10}}{z^9 (z - \alpha)}$$

to see that we have 9 poles at $z = 0$, and 9 roots of α^{10} as zeros:

$$z = \alpha e^{jn\pi/10} \qquad \text{for } n = 1, 2, \ldots, 9$$

As in the case of the running average filter, there is no pole or zero at $z = \alpha$, because $H_1(\alpha)$ is not zero and it is not infinity; in fact, $H_1(\alpha)$ is equal to 10. Finally, we plot these poles and zeros below for $\alpha = 1/2$:



This pole-zero plot can be generated with the MATLAB command called `zplane`, or the DSP-First function called `zzplane()`.

(d) There are two forms for the inverse system. The poles and zeros are plotted above.

$$H_2(z) = \frac{1}{H_1(z)} = \frac{1-\alpha z^{-1}}{1-\alpha^{10}z^{-10}}$$

$$\text{or,} \qquad H_2(z) = \frac{1}{\sum_{k=0}^{9} \alpha^k z^{-k}} = \frac{1}{1+\sum_{k=1}^{9} \alpha^k z^{-k}}$$

In the combined transfer function of $H(z) = H_1(z)H_2(z)$, for each pole we have a corresponding zero. As a result, the poles and zeros cancel each other, leaving no poles or zeros for $H(z)$, because $H(z) = 1$.

(e) There are two ways to write the difference equation for the second filter. In the first method, we use the system function:

$$H_2(z) = \frac{1}{\sum_{k=0}^{9} \alpha^k z^{-k}} = \frac{1}{1+\sum_{k=1}^{9} \alpha^k z^{-k}}$$

The denominator coefficients will be used as the feedback coefficients in the IIR filter:

$$y[n] = w[n] - \sum_{k=1}^{9} \alpha^k y[n-k]$$

In the second approach, we use the alternate form of the system function:

$$H_2(z) = \frac{1-\alpha z^{-1}}{1-\alpha^{10}z^{-10}}$$

and now when we acquire the IIR filter coefficients, we get:

$$y[n] = \alpha^{10} y[n-10] + w[n] - \alpha w[n-1]$$

It is interesting that these two systems give exactly the same output, even though they carry out different computations!

(f) As in part (e) there are two possibilities for the MATLAB code:

`bb = [1];` and `aa = [` $1\ \alpha\ \alpha^2\ \alpha^3\ \alpha^4\ \alpha^5\ \alpha^6\ \alpha^7\ \alpha^8\ \alpha^9$ `]`

or, `bb = [1,`$-\alpha$`];` and `aa = [1, zeros(1,9),` $-\alpha^{10}$`]`

In either case, the call to the `filter()` function is: `yy = filter( bb, aa, xx );`

PROBLEM 9.3:

See the solution to Problem 7.4 from Spring 1999 on Web-CT.

PROBLEM 9.4:

See the solution to Problem 7.5 from Spring 1999 on Web-CT.

PROBLEM 9.5:

See the solution to Problem 7.6 from Spring 1999 on Web-CT.