

EE-2025

Fall-99

Lecture 24

Sampling and Reconstruction

3-Dec-99

Info: Web-CT, Lab, HW

- **Calendar: Final Exams**
 - 11am section–Period 9, Weds, 12/15
 - 12pm section–Period 13, Fri, 12/17
 - **NO switching allowed**
- **Last Lab: Review Quiz on MATLAB**
- **Reading Assignment: Ch 13 of Notes.**
- **Prob Set #13 – due 3-Dec**
- **Prob Set #14**
 - **not handed in, but covered on final exam**

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LECTURE OBJECTIVES

- **Sampling Theorem Revisited**
 - **GENERAL: in the FREQUENCY DOMAIN**
 - Fourier transform of sampled signal
 - Reconstruction from samples
- **Review of FT properties**
 - Convolution <--> multiplication
 - Frequency shifting
 - Review of AM

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Table of Easy FT Properties

Linearity Property

$$ax_1(t) + bx_2(t) \Leftrightarrow aX_1(j\omega) + bX_2(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

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Table of FT Properties

Convolution Property

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

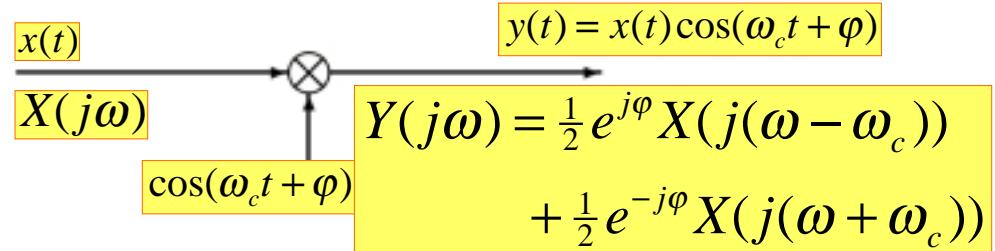
Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

Scaling Property

$$x(\beta t) \Leftrightarrow \frac{1}{|\beta|} X(j \frac{\omega}{\beta})$$

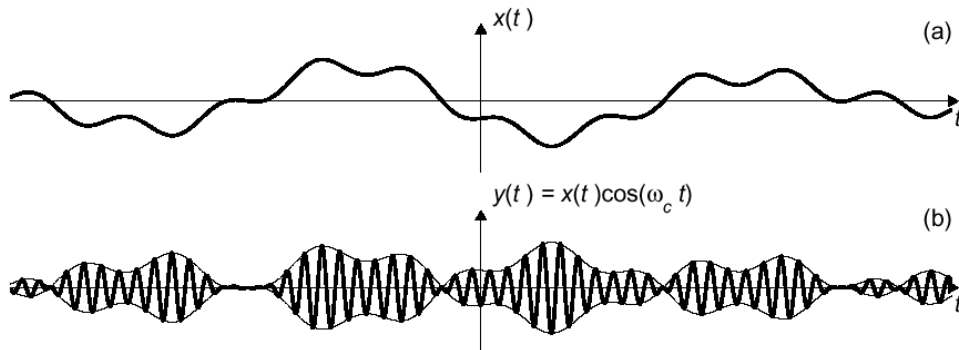
Amplitude Modulator



- $x(t)$ **modulates** the amplitude of the cosine wave. The result in the frequency-domain is two **SHIFTED** copies of $X(j\omega)$.

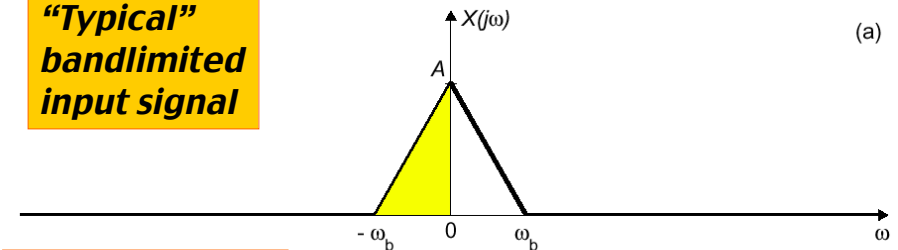
Double Sideband AM (Time-Domain)

- In the time-domain, the envelope of sinewave peaks follows $|x(t)|$

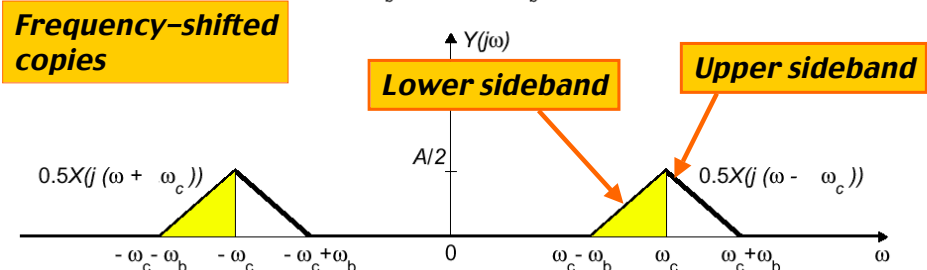


DSBAM: Frequency-Domain

“Typical” bandlimited input signal

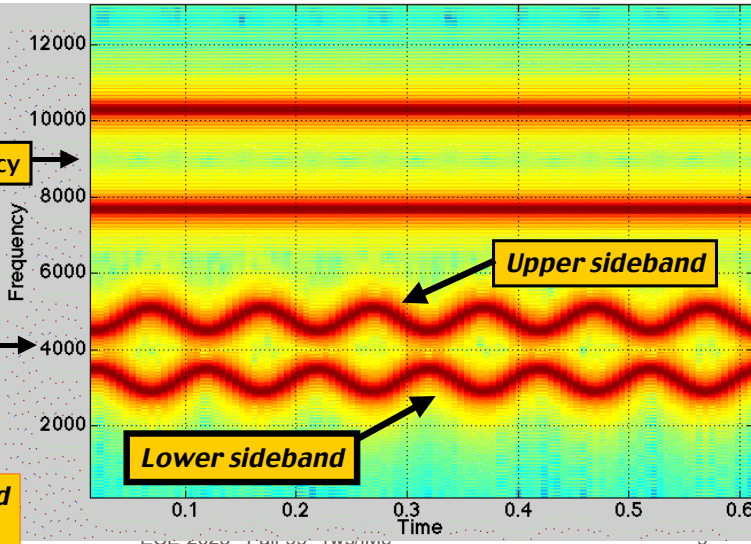


Frequency-shifted copies



DSBAM: Frequency-Domain

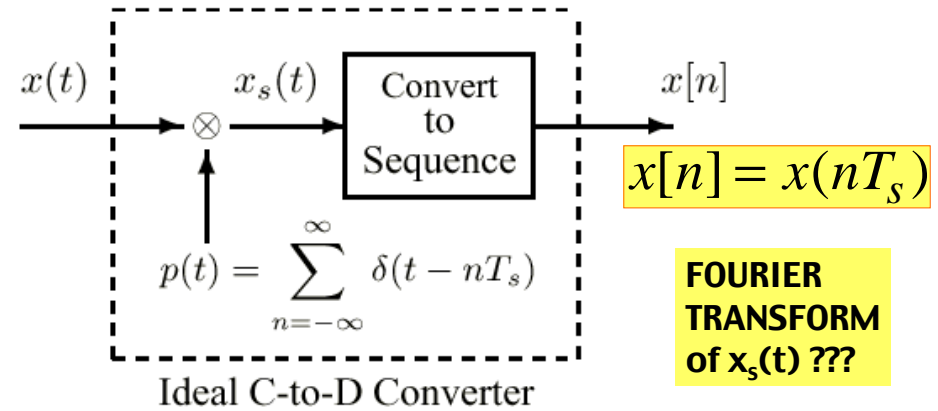
“Typical” bandlimited input signals



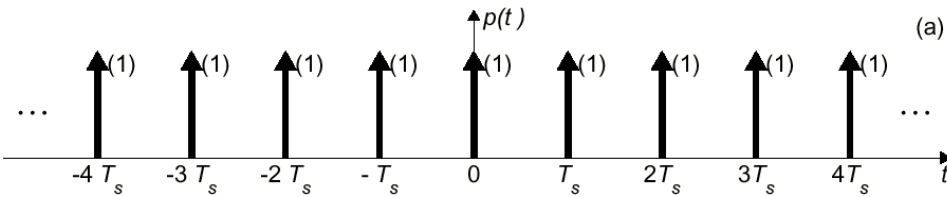
Frequency-shifted copies

Ideal C-to-D Converter

- Mathematical Model for A-to-D



Periodic Impulse Train

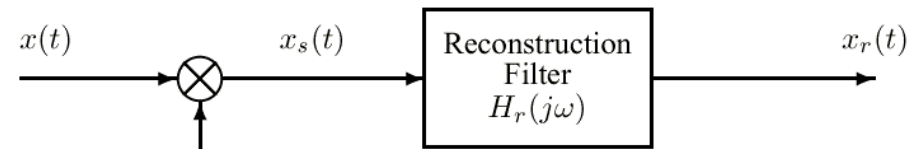


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T_s}$$

$$\omega_s = \frac{2\pi}{T_s}$$

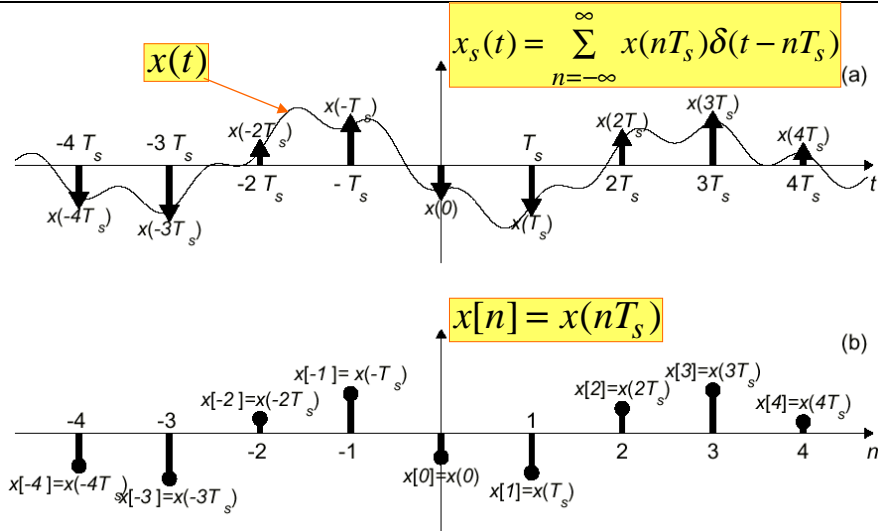
Impulse Train Sampling



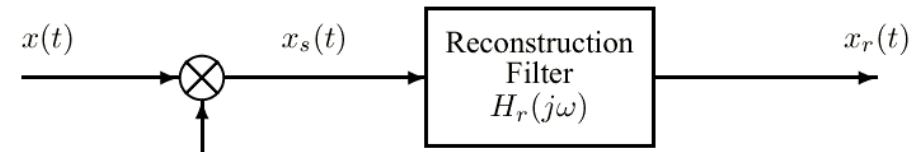
$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} \underline{x(t)\delta(t - nT_s)}$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} \underline{x(nT_s)\delta(t - nT_s)}$$

Illustration of Sampling



Sampling: Freq. Domain



$$p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

**EXPECT
FREQUENCY
SHIFTING**

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

Frequency-Domain Analysis

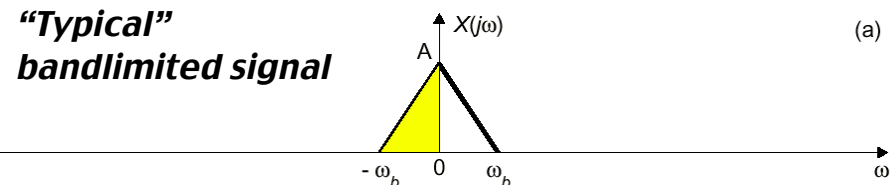
$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$x_s(t) = x(t) \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} x(t) e^{jk\omega_s t}$$

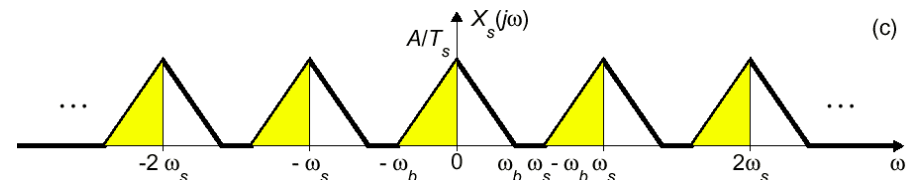
$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

Frequency-Domain Representation of Sampling

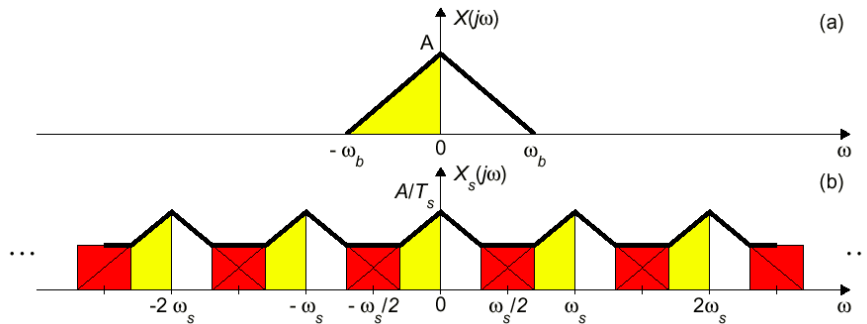


$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

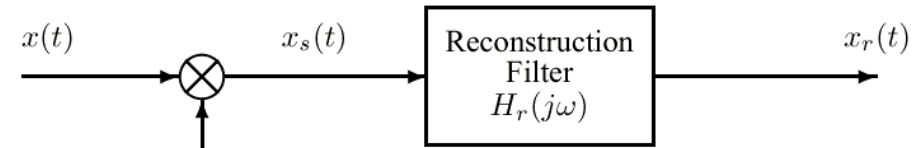


Aliasing Distortion

- If $\omega_s < 2\omega_b$, the copies of $X(j\omega)$ overlap, and we have **aliasing distortion**.



Reconstruction of $x(t)$

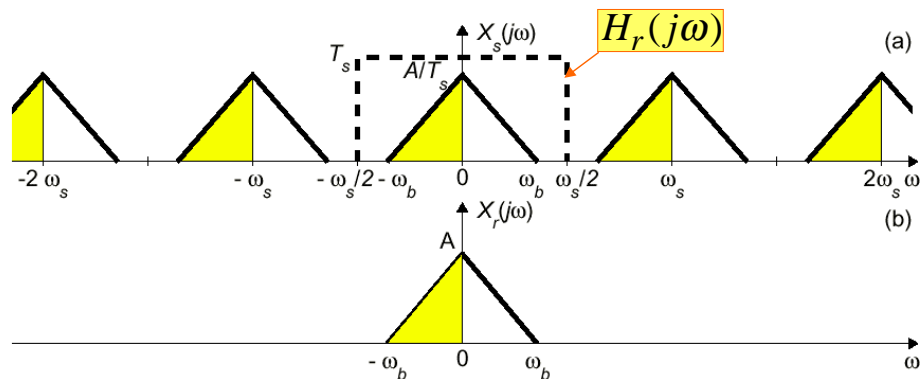


$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega)X_s(j\omega)$$

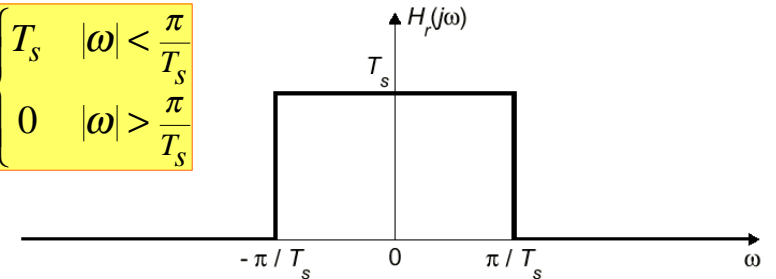
Reconstruction in the Frequency-Domain



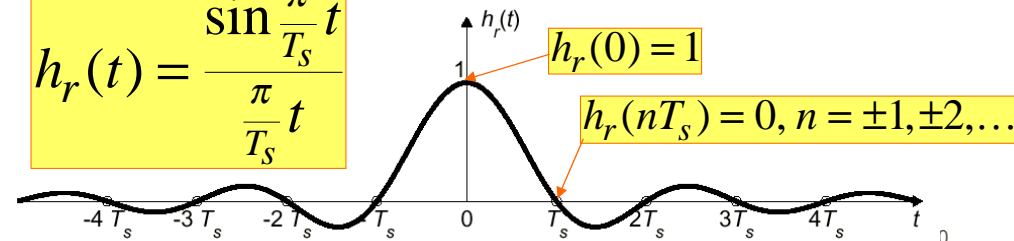
- If $\omega_s > 2\omega_b$, the copies of $X(j\omega)$ do not overlap, so $X_r(j\omega) = H_r(j\omega) X_s(j\omega)$.

Ideal Reconstruction Filter

$$H_r(j\omega) = \begin{cases} T_s & |\omega| < \frac{\pi}{T_s} \\ 0 & |\omega| > \frac{\pi}{T_s} \end{cases}$$



$$h_r(t) = \frac{\sin \frac{\pi}{T_s} t}{\frac{\pi}{T_s} t}$$



Signal Reconstruction

$$x_r(t) = h_r(t) * x_s(t) = h_r(t) * \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) h_r(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \frac{\pi}{T_s} (t - nT_s)}{\frac{\pi}{T_s} (t - nT_s)}$$

Ideal bandlimited interpolation formula

Shannon Sampling Theorem

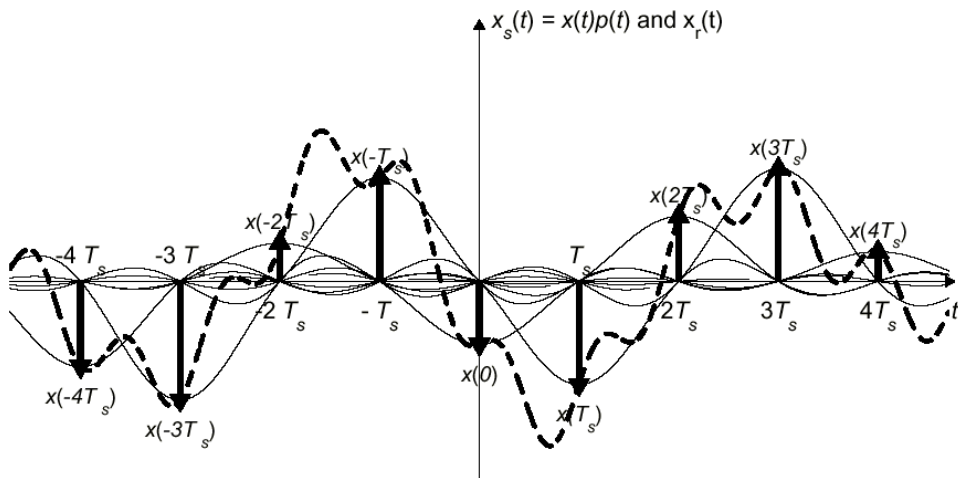
■ **“SINC” Interpolation** is the ideal

- PERFECT RECONSTRUCTION
- of BANDLIMITED SIGNALS

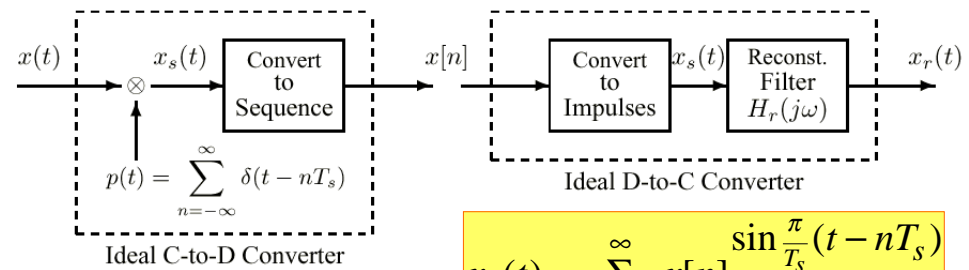
A signal $x(t)$ with bandlimited Fourier transform such that $X(j\omega) = 0$ for $|\omega| \geq \omega_b$ can be reconstructed exactly from samples taken with sampling rate $\omega_s = 2\pi/T_s \geq 2\omega_b$ using the following bandlimited interpolation formula:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \left[\frac{\pi}{T_s} (t - nT_s) \right]}{\frac{\pi}{T_s} (t - nT_s)}$$

Reconstruction in Time-Domain



Ideal C-to-D and D-to-C



$$x[n] = x(nT_s)$$

Ideal Sampler

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \frac{\pi}{T_s} (t - nT_s)}{\frac{\pi}{T_s} (t - nT_s)}$$

Ideal bandlimited interpolator

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega) X_s(j\omega)$$