EE-2025 Fall-99

Lecture 20

Frequency Response of Continuous-Time Systems

12-Nov-99

Info: Web-CT, Lab, HW

- Calendar:
  - Quiz #3 is 22-Nov
- CHECK YOUR GRADES !!!
  - Web-CT is the OFFICIAL gradebook
- Prob Set #11 is due today
- Lab #11 on SYMBOLIC FOURIER SERIES
  - Then 1 more Lab

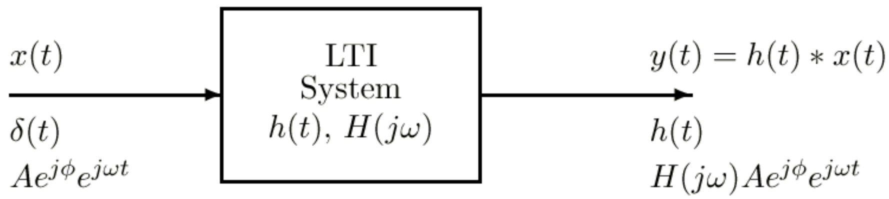
READING ASSIGNMENTS

- This Lecture:
  - Chapter 11, pp. 1100-1123
- Other Reading:
  - Recitation: Ch. 11, pp. 1128-1132
  - Next Lecture: Chapter 12, 1200-1230

LECTURE OBJECTIVES

- Review of convolution
  - THE operation for LTI Systems
- Complex exponential input signals
  - Frequency Response
  - Cosine signals
- Fourier Series
- Filtering

LTI Systems



- Convolution defines LTI system

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Response to complex exponential gives frequency response ***H(jω)***

Thought Process #1

- SUPERPOSITION
  - Make x(t) a weighted sum of signals
  - Then y(t) is also a sum—different weights
    - DIFFERENT OUTPUT SIGNALS usually
- Use SINUSOIDS
  - Make x(t) a weighted sum of sinusoids
  - Then y(t) is also a sum of sinusoids
    - Different Magnitudes and Phase
- LTI SYSTEMS: Sinusoidal Response

Thought Process #2

- SUPERPOSITION
  - Make x(t) a weighted sum of signals
- Use SINUSOIDS
  - Any x(t) = weighted sum of sinusoids
  - HOW? Use FOURIER ANALYSIS INTEGRAL
    - To find the weights from x(t)
- LTI SYSTEMS:
  - Frequency Response changes each sinusoidal component

Complex Exponential Input

$$x(t) = Ae^{j\phi}e^{j\omega t} \mapsto y(t) = H(j\omega)Ae^{j\phi}e^{j\omega t}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)Ae^{j\phi}e^{j\omega(t-\tau)}d\tau$$

$$y(t) = \left(\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau\right)Ae^{j\phi}e^{j\omega t}$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

Frequency Response

When does $H(j\omega)$ Exist?

- When is $|H(j\omega)| < \infty$?

$$|H(j\omega)| = \left| \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau \right| \le \int_{-\infty}^{\infty} |h(\tau)| |e^{-j\omega\tau}| d\tau$$

$$|H(j\omega)| \le \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

- Thus the frequency response exists if the LTI sytem is a stable system.

Ideal Delay: $y(t) = x(t - t_d)$

$$x(t) = e^{j\omega t} \mapsto$$

$$y(t) = e^{j\omega(t - t_d)} = e^{-j\omega t_d} e^{j\omega t}$$

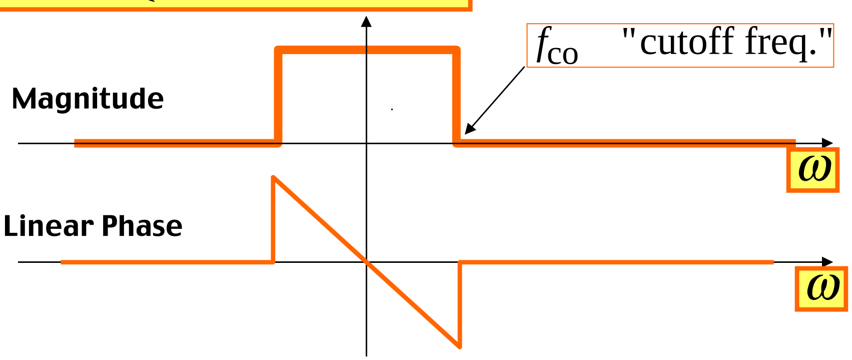
$H(j\omega)$

$$H(j\omega) = \int_{-\infty}^{\infty} \delta(\tau - t_d) e^{-j\omega\tau} d\tau = e^{-j\omega t_d}$$

$$H(j\omega) = e^{-j\omega t_d}$$

Ideal Lowpass Filter

$$H_{LP}(j\omega) = \begin{cases} e^{-j\omega t_d} & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$



Example: Ideal Low Pass

$$H_{LP}(j\omega) = \begin{cases} e^{-j3\omega} & |\omega| < 2 \\ 0 & |\omega| > 2 \end{cases}$$

$$x(t) = 10 e^{j\pi/3} e^{jt} \mapsto y(t) = H(j1) 10 e^{j\pi/3} e^{jt}$$

$$y(t) = e^{-j3} 10 e^{j\pi/3} e^{jt} = 10 e^{j\pi/3} e^{j(t-3)}$$

Cosine Input

$$x(t) = A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

$$y(t) = H(j\omega_0)\frac{A}{2}e^{j\phi}e^{j\omega_0 t} + H(-j\omega_0)\frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

Since $H(-j\omega_0) = H^*(j\omega_0)$

$$y(t) = A|H(j\omega_0)|\cos(\omega_0 t + \phi + \angle H(j\omega_0))$$

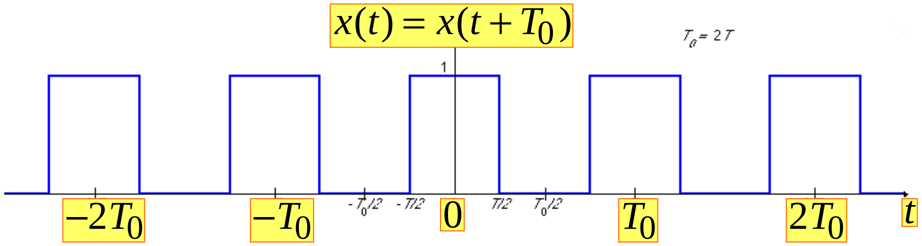
STRATEGY

- **ANALYSIS**
  - **Get representation from the signal**
  - **Works for PERIODIC Signals**
- **Fourier Series**
  - **INTEGRAL over one period**

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}dt$$

General Periodic Signals

$x(t) = x(t + T_0)$

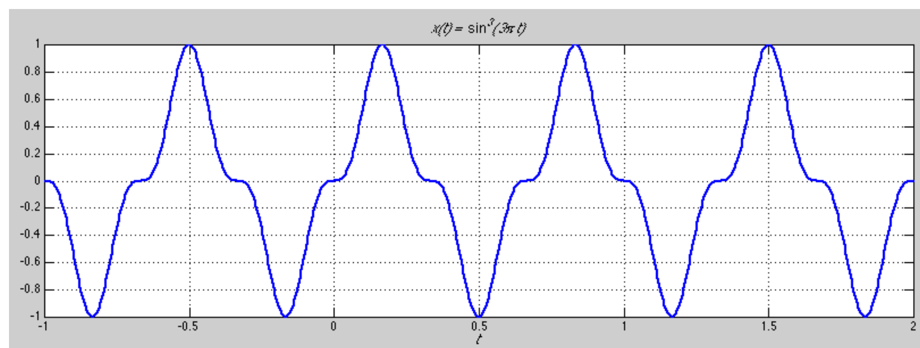


$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$ ← *Fourier Synthesis*

Fundamental Freq. $\omega_0 = 2\pi / T_0 = 2\pi f_0$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}dt$$ ← *Fourier Analysis*

Example $x(t) = \sin^3(3\pi t)$

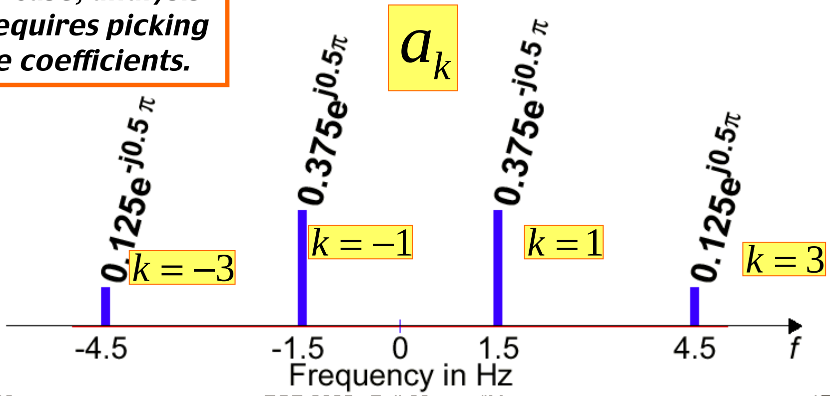


$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

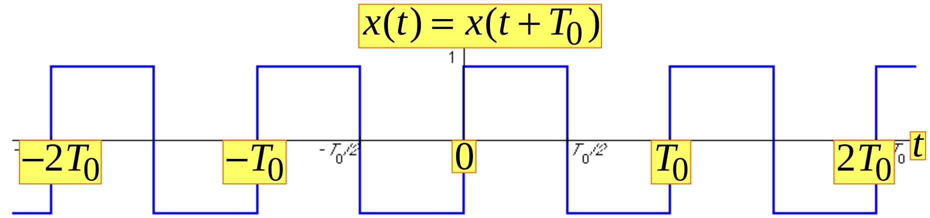
Example $x(t) = \sin^3(3\pi t)$

$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

In this case, analysis just requires picking off the coefficients.



Square Wave Signal



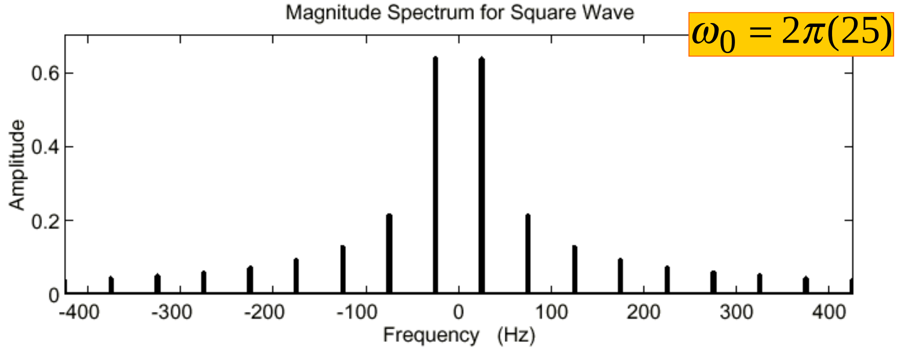
$$a_k = \frac{1}{T_0}\int_0^{T_0/2}(1)e^{-j\omega_0 kt}dt + \frac{1}{T_0}\int_{T_0/2}^{T_0}(-1)e^{-j\omega_0 kt}dt$$

$$a_k = \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_0^{T_0/2} - \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_{T_0/2}^{T_0} = \frac{1-e^{-j\pi k}}{j\pi k}$$

Spectrum from Fourier Series

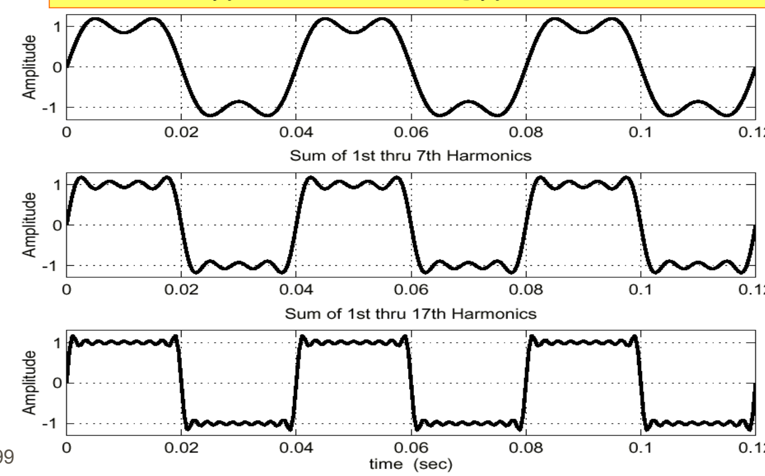
$$a_k = \frac{1-e^{-j\pi k}}{j\pi k} = \begin{cases} \dfrac{2}{j\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = 0, \pm2, \pm4, \ldots \end{cases}$$

Magnitude Spectrum for Square Wave

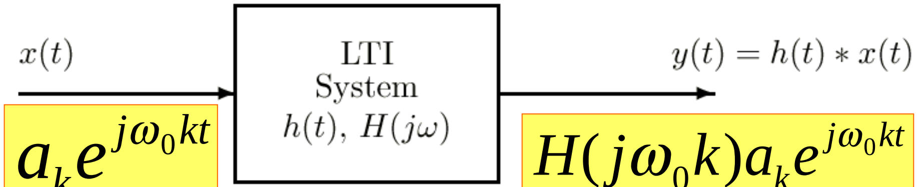


Fourier Synthesis

$$x_N(t) = \frac{4}{\pi}\sin(\omega_0 t) + \frac{4}{3\pi}\sin(3\omega_0 t) + \ldots$$



LTI Systems with Periodic Inputs



$a_k e^{j\omega_0 kt}$ $\mapsto$ $H(j\omega_0 k) a_k e^{j\omega_0 kt}$

Output has *same* frequencies

- **By superposition,**

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k) e^{j\omega_0 kt} = \sum_{k=-\infty}^{\infty} b_k e^{j\omega_0 kt}$$

$$b_k = a_k H(j\omega_0 k)$$

Example

$$H(j\omega) = e^{-j\omega t_d}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \mapsto y(t) = \sum_{k=-\infty}^{\infty} b_k e^{j\omega_0 kt}$$

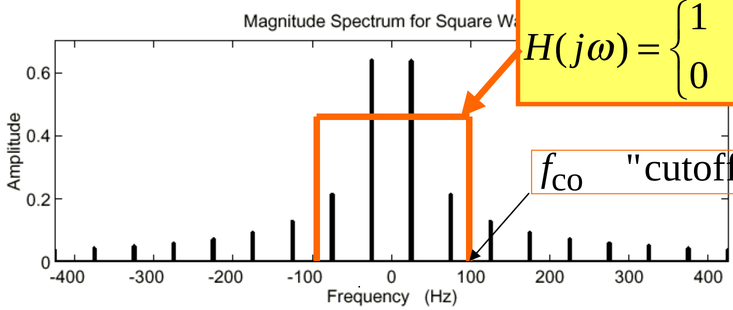
$$b_k = a_k H(j\omega_0 k) = a_k e^{-j\omega_0 k t_d}$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k e^{-j\omega_0 k t_d} e^{j\omega_0 kt} = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k(t-t_d)}$$

$$\therefore \quad y(t) = x(t - t_d)$$

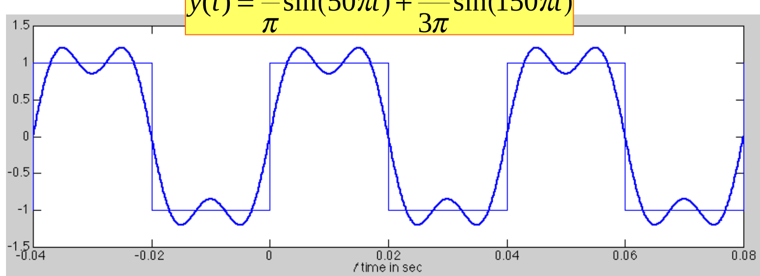
Ideal Lowpass Filter

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$



f_{co} "cutoff freq."

$$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t)$$



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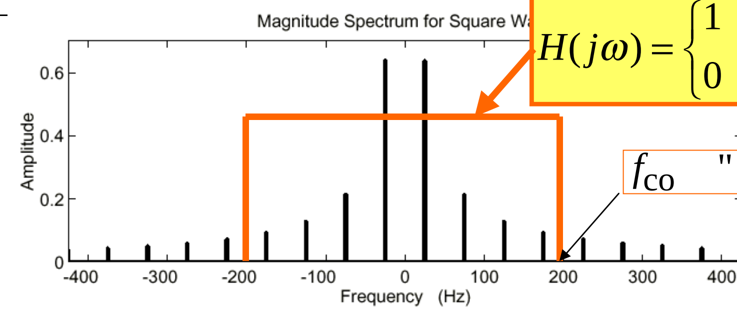

f_{co} "cutoff"

$$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t) + \frac{4}{5\pi}\sin(250\pi t) + \frac{4}{7\pi}\sin(350\pi t)$$