

**EE-2025**

**Fall-99**

**Lecture 12**

**Z Transforms: Introduction**

**8-Oct-99**

**Info: Web-CT, Lab, HW**

- **Quiz #2 on 25-Oct (Monday)**
  - Coverage: up to HW #8
- **Prob Set #6 due TODAY**
  - Solutions for #6 will be posted soon
  - On-Line HW #5 due before Sunday 6AM
- **Lab #7 on Bandpass Filters**
  - Frequency Response
  - **NO Lab the week of 18-Oct**

**READING ASSIGNMENTS**

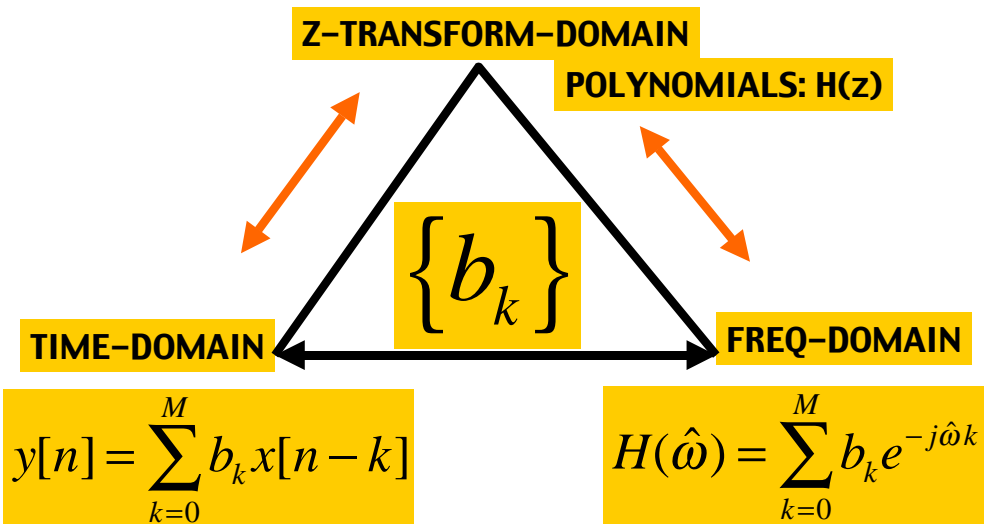
- **This Lecture:**
  - Chapter 7, pp. 202–216
- **Other Reading:**
  - Recitation: Ch. 7, pp. 217–220
    - CASCADING SYSTEMS
  - Next Lecture: Chapter 7, more

**LECTURE OBJECTIVES**

- **INTRODUCE the Z-TRANSFORM**
  - Give Mathematical Definition
  - Show how **H(z) POLYNOMIAL** simplifies analysis
    - **CONVOLUTION** EXAMPLE
- **Z-Transform can be applied to**
  - FIR Filter:  $h[n] \rightarrow H(z)$
  - Signals:  $x[n] \rightarrow X(z)$

$$H(z) = \sum_n h[n] z^{-n}$$

# TWO (no, THREE) DOMAINS



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# TRANSFORM CONCEPT

- Move to a new domain where
  - OPERATIONS are EASIER/FAMILIAR
  - Use **POLYNOMIALS**
- TRANSFORM both ways
  - $x[n] \rightarrow X(z)$  (into the z domain)
  - $X(z) \rightarrow x[n]$  (back to the time domain)

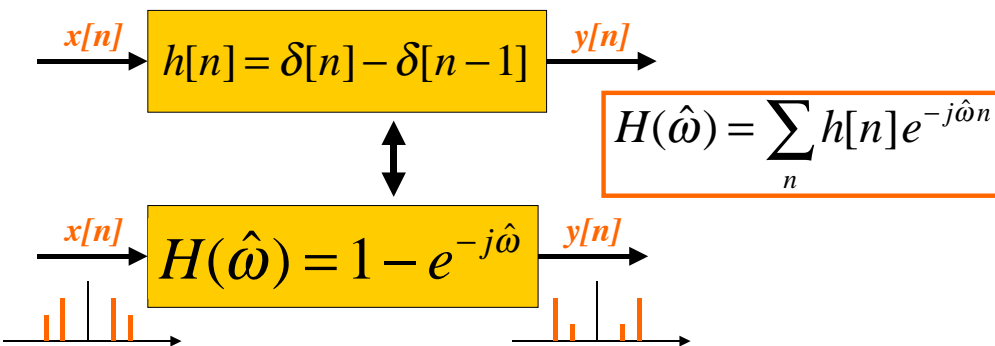
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# “TRANSFORM” EXAMPLE

## Equivalent Representations



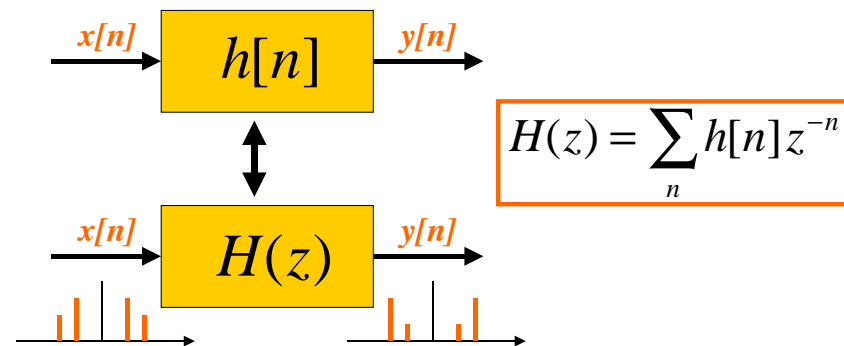
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# Z-TRANSFORM IDEA

## POLYNOMIAL REPRESENTATION



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# Z-Transform DEFINITION

■ POLYNOMIAL Representation of LTI SYSTEM:

$$H(z) = \sum_n h[n]z^{-n}$$

APPLIES to Any SIGNAL

■ EXAMPLE:

$$\{h[n]\} = \{2, 0, -3, 0, 2\}$$

$$\begin{aligned} H(z) &= 2z^{-0} + 0z^{-1} - 3z^{-2} + 0z^{-3} + 2z^{-4} \\ &= 2 - 3z^{-2} + 2z^{-4} \\ &= 2 - 3(z^{-1})^2 + 2(z^{-1})^4 \end{aligned}$$

POLYNOMIAL in  $z^{-1}$

# Z-Transform EXAMPLE

■ ANY SIGNAL has a z-Transform:

$$X(z) = \sum_n x[n]z^{-n}$$

Example 7.1

$n$	$n < -1$	-1	0	1	2	3	4	5	$n > 5$
$x[n]$	0	0	2	4	6	4	2	0	0

$$X(z) = ? \quad X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4}$$

Example 7.2

$$X(z) = 1 - 2z^{-1} + 3z^{-3} - z^{-5}$$

EXPONENT GIVES TIME LOCATION

$$x[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ -2 & n = 1 \\ 0 & n = 2 \\ 3 & n = 3 \\ 0 & n = 4 \\ -1 & n = 5 \\ 0 & n > 5 \end{cases}$$

$$x[n] = ?$$

$$x[n] = \delta[n] - 2\delta[n-1] + 3\delta[n-3] - \delta[n-5]$$

# Z-Transform of FIR Filter

■ CALLED the SYSTEM FUNCTION

■  $h[n]$  is same as  $\{b_k\}$

SYSTEM FUNCTION

$$H(z) = \sum_{k=0}^M b_k z^{-k} = \sum_{k=0}^M h[k] z^{-k}$$

$$y[n] = \sum_{k=0}^M b_k x[n-k] = \sum_{k=0}^M h[k] x[n-k]$$

FIR DIFFERENCE EQUATION

CONVOLUTION

## Z-Transform of FIR Filter

- Get  $H(z)$  DIRECTLY from the  $\{b_k\}$
- Example 7.3 in the book:

$$y[n] = 6x[n] - 5x[n-1] + x[n-2]$$

$$\{b_k\} = \{6, -5, 1\}$$

$$H(z) = \sum b_k z^{-k} = 6 - 5z^{-1} + z^{-2}$$

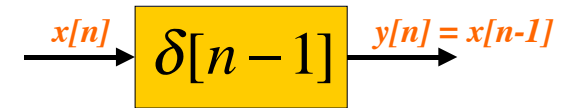
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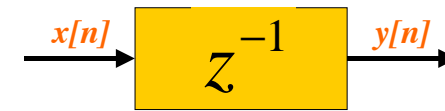
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## Ex. DELAY SYSTEM

- UNIT DELAY: find  $h[n]$  and  $H(z)$



$$H(z) = \sum \delta[n-1] z^{-n} = z^{-1}$$



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## DELAY EXAMPLE

- UNIT DELAY: find  $y[n]$  via polynomials
- $x[n] = \{3, 1, 4, 1, 5, 9, 0, 0, 0, \dots\}$

$$Y(z) = z^{-1} X(z)$$

$$Y(z) = z^{-1} (3 + z^{-1} + 4z^{-2} + z^{-3} + 5z^{-4} + 9z^{-5})$$

$$Y(z) = 0z^0 + 3z^{-1} + z^{-2} + 4z^{-3} + z^{-4} + 5z^{-5} + 9z^{-6}$$

$n$	$n < 0$	0	1	2	3	4	5	6	$n > 6$
$y[n]$	0	0	3	1	4	1	5	9	0

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## DELAY PROPERTY

A delay of one sample multiplies the  $z$ -transform by  $z^{-1}$ .

$$x[n-1] \iff z^{-1} X(z)$$

Time delay of  $n_0$  samples multiplies the  $z$ -transform by  $z^{-n_0}$

$$x[n-n_0] \iff z^{-n_0} X(z)$$

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# GENERAL I/O PROBLEM

■ Input is  $x[n]$ , find  $y[n]$  (for FIR,  $h[n]$ )

■ How to combine  $X(z)$  and  $H(z)$  ?

Example 7.5

$$x[n] = \delta[n - 1] - \delta[n - 2] + \delta[n - 3] - \delta[n - 4]$$

$$\text{and } h[n] = \delta[n] + 2\delta[n - 1] + 3\delta[n - 2] + 4\delta[n - 3]$$

$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$

$$\text{and } H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

# FIR Filter = CONVOLUTION

$x[n], X(z)$	0	+1	-1	+1	-1			
$h[n], H(z)$	1	2	3	4				
-----								
	0	+1	-1	+1	-1			
		0	+2	-2	+2	-2		
			0	+3	-3	+3	-3	
				0	+4	-4	+4	-4
-----								
$y[n], Y(z)$	0	+1	+1	+2	+2	-3	+1	-4

$$y[n] = \sum_{k=0}^M b_k x[n - k] = \sum_{k=0}^M h[k] x[n - k]$$

# CONVOLUTION PROPERTY

■ PROOF:

$$y[n] = x[n] * h[n] = \sum_{k=0}^M h[k] x[n - k]$$

$$Y(z) = \sum_{k=0}^M h[k] (z^{-k} X(z))$$

MULTIPLY  
Z-TRANSFORMS

$$= \left( \sum_{k=0}^M h[k] z^{-k} \right) X(z) = H(z) X(z).$$

# CONVOLUTION EXAMPLE

■ MULTIPLY the z-TRANSFORMS:

Example 7.5

$$x[n] = \delta[n - 1] - \delta[n - 2] + \delta[n - 3] - \delta[n - 4]$$

$$\text{and } h[n] = \delta[n] + 2\delta[n - 1] + 3\delta[n - 2] + 4\delta[n - 3]$$

$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$

$$\text{and } H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

MULTIPLY  $H(z)X(z)$

# CONVOLUTION EXAMPLE

■ Finite-Length input  $x[n]$

■ FIR Filter (L=4)

MULTIPLY  
Z-TRANSFORMS

$$\begin{aligned}
 Y(z) &= H(z)X(z) \\
 &= (1 + 2z^{-1} + 3z^{-2} + 4z^{-3})(z^{-1} - z^{-2} + z^{-3} - z^{-4}) \\
 &= z^{-1} + (-1 + 2)z^{-2} + (1 - 2 + 3)z^{-3} + (-1 + 2 - 3 + 4)z^{-4} \\
 &\quad + (-2 + 3 - 4)z^{-5} + (-3 + 4)z^{-6} + (-4)z^{-7} \\
 &= z^{-1} + z^{-2} + 2z^{-3} + 2z^{-4} - 3z^{-5} + z^{-6} - 4z^{-7}
 \end{aligned}$$

$y[n] = ?$

# CASCADE SYSTEMS

■ Does the order of  $S_1$  &  $S_2$  matter?

■ NO, LTI SYSTEMS can be rearranged !!!

■ Remember:  $h_1[n] * h_2[n]$

■ How to combine  $H_1(z)$  and  $H_2(z)$  ?

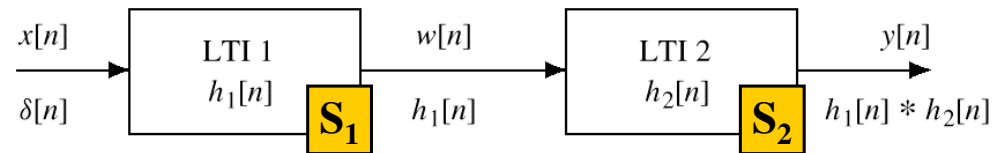
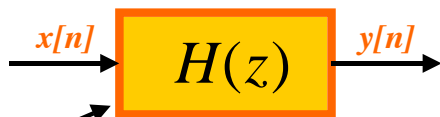
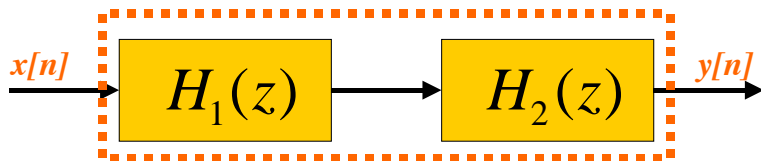


Figure 5.19 A Cascade of Two LTI Systems.

# CASCADE EQUIVALENT

■ Multiply the System Functions



EQUIVALENT  
SYSTEM

$$H(z) = H_1(z)H_2(z)$$