EE-2200 Winter-99

Lecture 7

Sampling & Aliasing

5-Feb-99

Information

- Check the Bulletin Board for msgs
  - Notes file: jesunotes.m
  - Spectrogram image display info
    - New M-file: plotspec.m
  - FORMAL Lab Report
- Problem Set #4 out today
- Quiz #2 on 1-March (Monday)

READING ASSIGNMENTS

- This Lecture:
  - Chapter 4, pp. 83-94
- Other Reading:
  - Recitation: Chapter 4, pp. 90-100
    - Strobe Demo
  - Next Lecture: Chap. 4, pp. 100-111

LECTURE OBJECTIVES

- SAMPLING can cause ALIASING
  - Sampling Theorem
  - Sampling Rate > 2(Highest Frequency)
- Spectrum for digital signals, x[n]
  - Normalized Frequency

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}$$

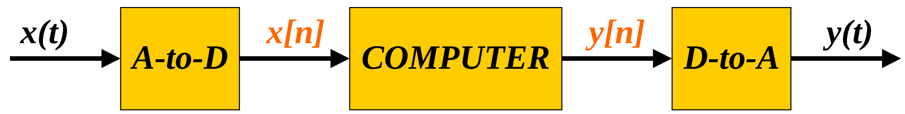
SYSTEMS Process Signals

- PROCESSING GOALS:
  - Change x(t) into y(t)
    - For example, more BASS
  - Improve x(t), e.g., image deblurring
  - Extract Information from x(t)

System IMPLEMENTATION

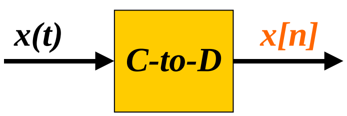
- ANALOG/ELECTRONIC:
  - Circuits: resistors, capacitors, op-amps

- DIGITAL/MICROPROCESSOR
  - Convert x(t) to numbers stored in memory



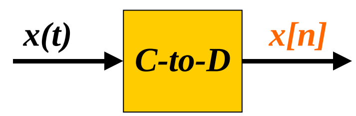
SAMPLING x(t)

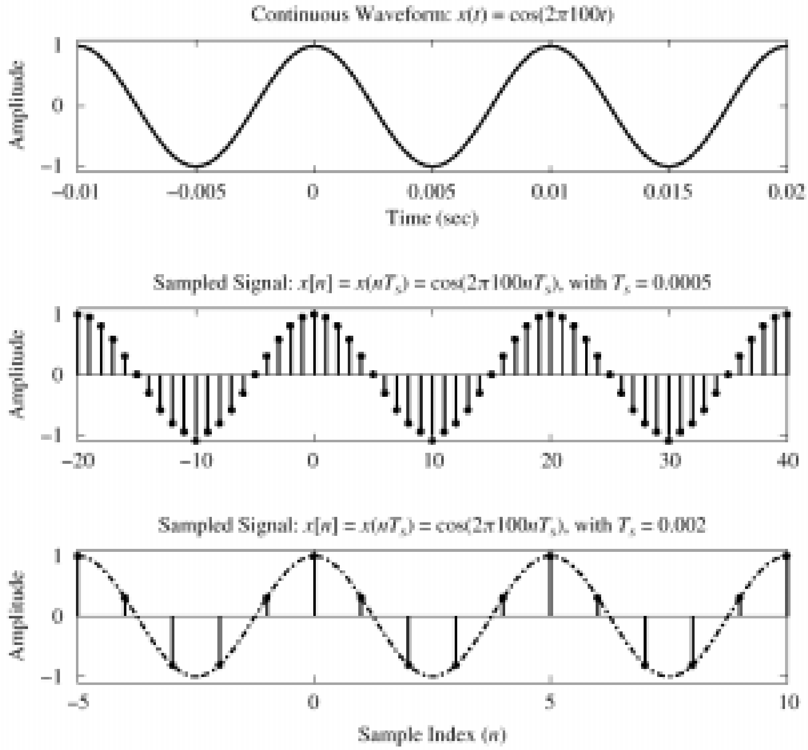
- SAMPLING PROCESS
  - Convert x(t) to numbers x[n]
  - "n" is an integer; x[n] is a sequence
  - "n" is the storage address in memory
- UNIFORM SAMPLING at $t = nT_s$
  - IDEAL: $x[n] = x(nT_s)$



SAMPLING RATE

- SAMPLING RATE (f_s)
  - $1/T_s$ = NUMBER of SAMPLES PER SECOND
  - 125 microsec ---> 8000 samples/sec
    - UNITS ARE HERTZ: 8000 Hz
- UNIFORM SAMPLING at $t = nT_s = n/f_s$
  - IDEAL: $x[n] = x(nT_s)=x(n/f_s)$





SAMPLING A SINUSOID

- **HOW OFTEN ?**
  - **DEPENDS on FREQUENCY of SINUSOID**
  - **ANSWERED by SHANNON/NYQUIST**
  - **DEPENDS on "RECONSTRUCTION"**

Shannon Sampling Theorem

A continuous-time signal $x(t)$ with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{max}$.

STORING DIGITAL SOUND

- **$x[n]$ is a SAMPLED SINUSOID**
  - **A list of numbers stored in memory**
- **CD rate is 44,100 samples per second**
- **16-bit samples**
- **Stereo uses 2 channels**
- **Number of bytes for 1 minute is**
  - **2 X (16/8) X 60 X 44100 = 10.584 Mbytes**

DISCRETE-TIME SINUSOID

- **Change x(t) into x[n]**

$$x(t) = A\cos(\omega t + \varphi)$$

$$x[n] = x(nT_s) = A\cos(\omega nT_s + \varphi)$$

$$x[n] = A\cos((\omega T_s)n + \varphi)$$

$$x[n] = A\cos(\hat{\omega} n + \varphi)$$

$$\hat{\omega} = \omega T_s$$

DIGITAL FREQUENCY

DIGITAL FREQUENCY

- DIGITAL FREQUENCY is NORMALIZED
- UNITS are radians, not rad/sec
- GOES from 0 to 2π, as f goes from 0 to the sampling frequency

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}$$

ALIASING DERIVATION

$$x(t) = A\cos(2\pi f_0 t + \phi) \qquad (4.1.4)$$

If we sample $x(t)$ with a sampling period of T_s, we get the sequence $x[n]$ with values

$$x[n] = x(nT_s) = A\cos(2\pi f_0 n T_s + \phi) \qquad (4.1.5)$$

Now consider another sinusoid with the same amplitude and phase, but with frequency $f_0 + \ell f_s$, where ℓ is an integer and $f_s = 1/T_s$.

$$y(t) = A\cos(2\pi(f_0 + \ell f_s)t + \phi)$$

If this second waveform, $y(t)$, is sampled with period T_s, we get

$$\begin{aligned} y[n] = y(nT_s) &= A\cos(2\pi(f_0 + \ell f_s)nT_s + \phi) \\ &= A\cos(2\pi f_0 n T_s + 2\pi \ell f_s T_s + \phi) \\ &= A\cos(2\pi f_0 n T_s + 2\pi \ell + \phi) \\ &= A\cos(2\pi f_0 n T_s + \phi) \\ &= x[n] \end{aligned}$$

ALIASING

- ADDING f_s or $2f_s$ TO THE FREQ of x(t)
- SAMPLED SIGNAL: $x[n] = x(n/f_s)$
  - x[n] STAYS THE SAME
- CAN'T TELL f_o FROM $(f_o + f_s)$ or $(f_o + 2f_s)$
- CALLED ALIASING

NORMALIZED FREQUENCY

- DIGITAL FREQUENCY

Normalized Radian Frequency

$$\hat{\omega} = \omega T_s$$

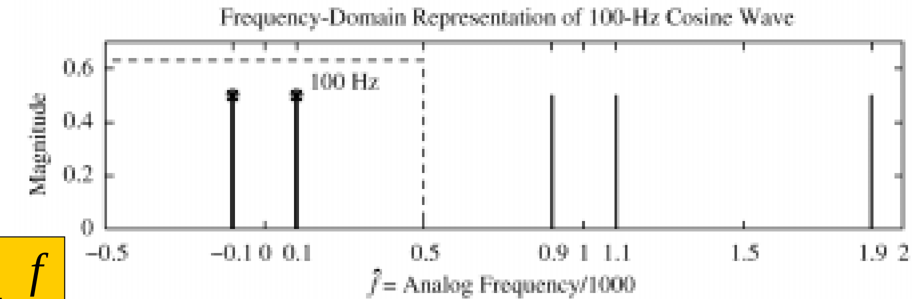
Normalized Cyclic Frequency

$$\hat{f} = \hat{\omega}/(2\pi) = f T_s = f/f_s$$

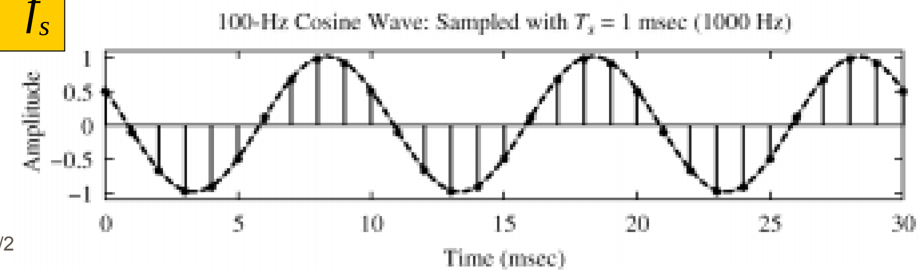
SPECTRUM for x[n]

- PLOT versus NORMALIZED FREQUENCY
- INCLUDE **ALL** SPECTRUM LINES
  - ALIASES
  - FOLDED ALIASES
    - ALIASES of NEGATIVE FREQS

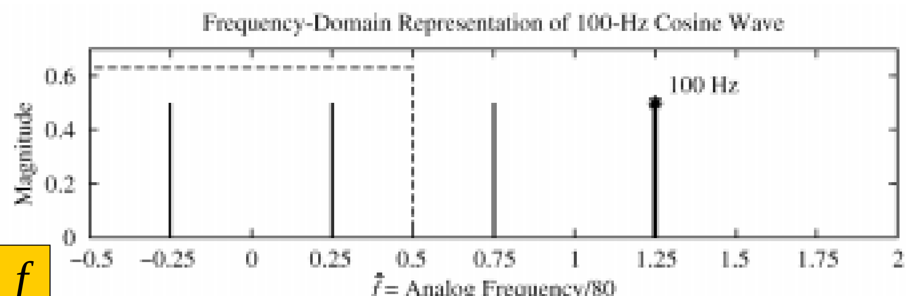
SPECTRUM (DIGITAL)



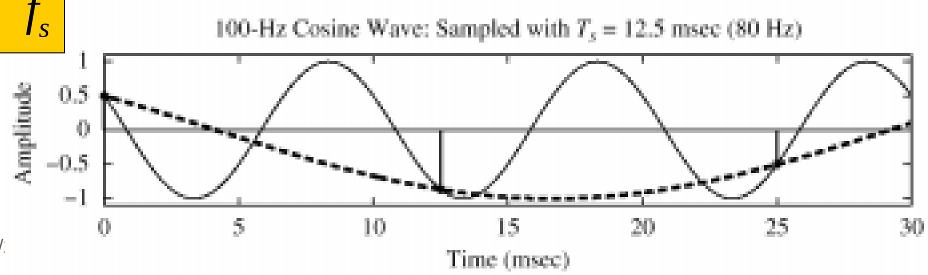
$$\hat{f} = \frac{f}{f_s}$$



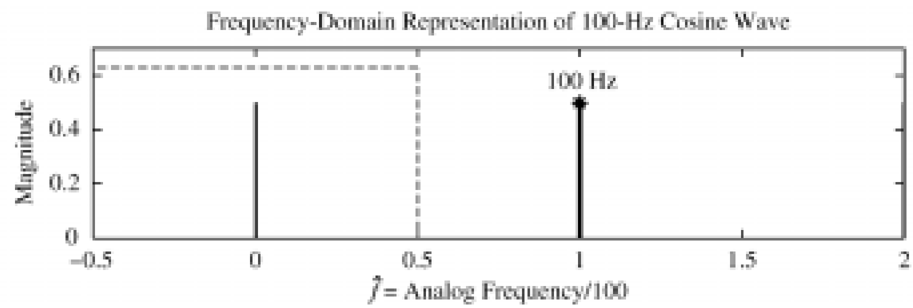
SPECTRUM of x[n]
ALIASING CASE

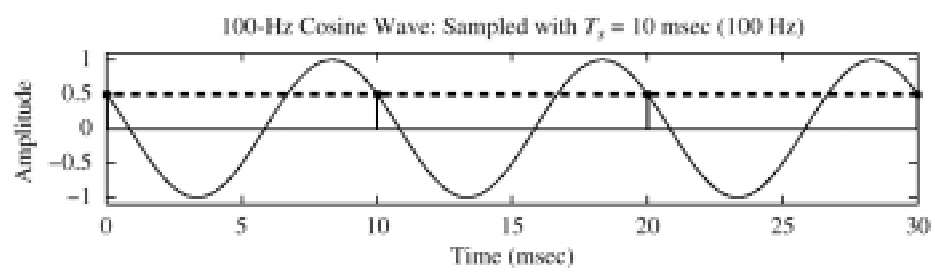


$$\hat{f} = \frac{f}{f_s}$$



SPECTRUM of x[n]
ALIASING to ZERO FREQ

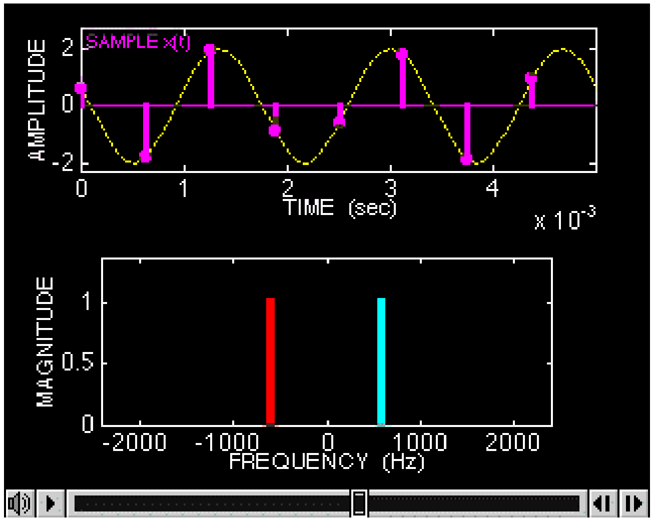




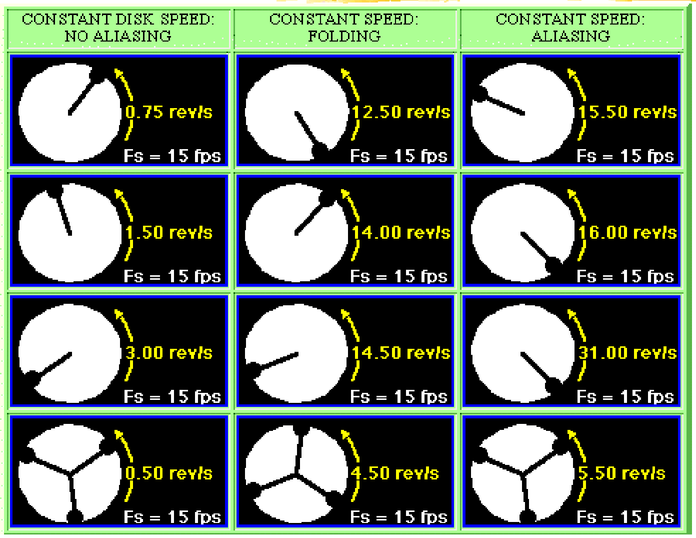
DEMOS from CHAPTER 4

- CD-ROM DEMOS
- SAMPLING DEMO
  - Different Sampling Rates
    - Aliasing of a Sinusoid
- STROBE DEMO
  - Synthetic vs. Real
  - Television SAMPLES at 30 fps
- Sampling & Reconstruction

SAMPLING DEMO



STROBE DEMO (Synthetic)



FOLDING DERIVATION

A second source of aliased signals actually comes from the negative frequency component of the cosine wave. These frequencies are $-f_0 + \ell f_s$, where ℓ is a positive or negative integer. Consider a third signal

$$w(t) = A\cos(2\pi(-f_0 + \ell f_s)t - \phi)$$

whose initial phase is the negative of that in (4.1.4). If we sample $w(t)$ with a sampling period of T_s, we now get

$$\begin{aligned} w[n] = w(nT_s) &= A\cos(2\pi(-f_0 + \ell f_s)nT_s - \phi) \\ &= A\cos(-2\pi f_0 nT_s + 2\pi\ell f_s T_s - \phi) \\ &= A\cos(-2\pi f_0 nT_s + 2\pi\ell - \phi) \\ &= A\cos(2\pi f_0 nT_s + \phi) \\ &= x[n] \end{aligned}$$

The fourth line in this equation is true because the cosine function is an even function; i.e., $\cos(-\theta) = \cos\theta$.

FOLDING

- ANOTHER x[n] THAT IS IDENTICAL
- CAN'T TELL f_o FROM $(f_s - f_o)$
  - Or, $(2f_s - f_o)$ or, $(3f_s - f_o)$
- EXAMPLE:
  - y(t) has 1000 Hz component
  - SAMPLING FREQ = 1500 Hz
  - WHAT is the "FOLDED" ALIAS ?

FOLDING DIAGRAM

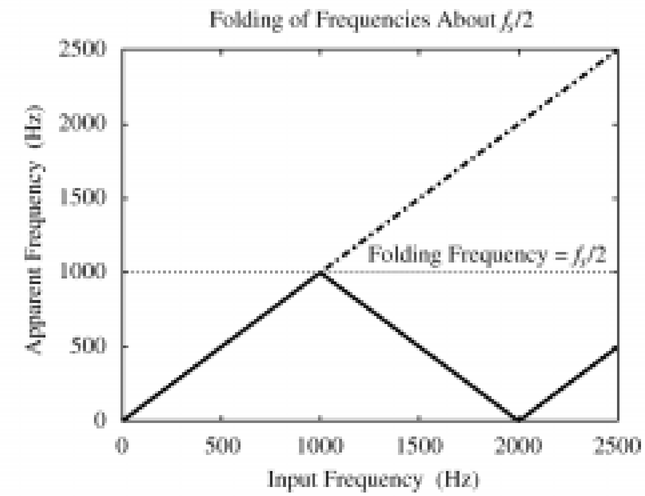


Figure 4.4 Folding of a sinusoid sampled at $f_s = 2000$ samples/sec. The apparent frequency is the lowest frequency of a sinusoid that has exactly the same samples as the input sinusoid.

SPECTRUM of x[n] FOLDING CASE