EE-2200 Winter-99

Lecture 14

IIR Filters: Feedback

5-March-99

Info: Web-CT, Lab, HW

- Calendar:
  - Final Exam is Period 11 (middle–Thurs)
  - Quiz Solutions are posted
- Grade Weightings will be posted
- Prob Set #7 is posted Monday
  - Prob–Set #7 due on last day
- FORMAL Lab Report Template

READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, pp. 249–263
  - Chapter 7, pp. 220–230
- Other Reading:
  - Recitation: Ch. 8, pp. 261–272
    - POLES & ZEROS
  - Next Lecture: Chapter 8, pp. 269–282

LECTURE OBJECTIVES

- INFINITE IMPULSE RESPONSE FILTERS
  - Define IIR Filters
  - Have FEEDBACK: PREVIOUS OUTPUTS

$$y[n] = \sum_{\ell=1}^{N} a_\ell y[n-\ell] + \sum_{k=0}^{M} b_k x[n-k]$$

  - Show how to compute the output y[n]
    - FIRST–ORDER CASE (N=1)
    - h[n] <--> H(z)

RECITATION (Previous)

- Relate H(z) to FREQUENCY RESPONSE

$$H(\hat{\omega}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

- THREE DOMAINS:
  - Show Relationship for FIR:

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

Z-Transform DEFINITION

- POLYNOMIAL Representation

$$H(z) = \sum_n h[n] z^{-n}$$

APPLIES to Any SIGNAL

- EXAMPLE:

$$\{h[n]\} = \{2, 0, -3, 0, 2\}$$

$$H(z) = 2z^{-0} + 0z^{-1} - 3z^{-2} + 0z^{-3} + 2z^{-4}$$

$$= 2 - 3z^{-2} + 2z^{-4}$$

Z-Transform of FIR Filter

- h[n] is same as $\{b_k\}$

SYSTEM FUNCTION

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \sum_{k=0}^{M} h[k] z^{-k}$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k] x[n-k]$$

FIR DIFFERENCE EQUATION

CONVOLUTION

CONVOLUTION PROPERTY

- Convolution in the n–domain
  - SAME AS
- Multiplication in the z–domain

$$y[n] = h[n] * x[n] \iff Y(z) = H(z)X(z)$$

MULTIPLY Z–TRANSFORMS

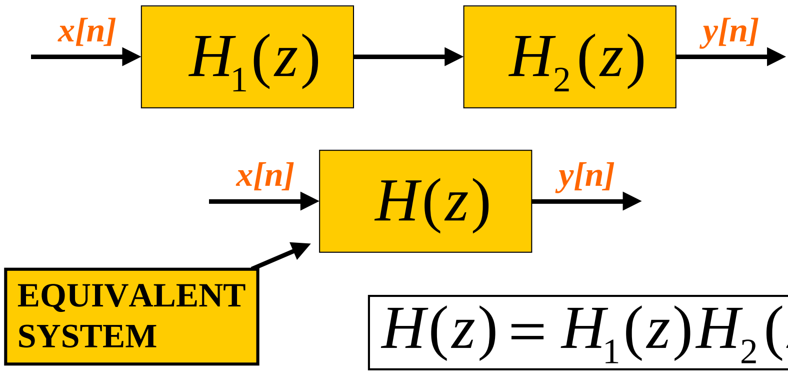
$$y[n] = x[n] * h[n]$$

$$= \sum_{k=0}^{M} h[k] x[n-k]$$

FIR Filter

CASCADE EQUIVALENT

- Multiply the System Functions



$$H(z) = H_1(z)H_2(z)$$

FREQUENCY RESPONSE ?

- Same Form: $z = e^{j\hat{\omega}}$

$\hat{\omega}$-Domain	$\Longleftrightarrow$	z-Domain
$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$	$\Longleftrightarrow$	$H(z) = \sum_{k=0}^{M} b_k z^{-k}$

SAME COEFFICIENTS

CHANGE in NOTATION

- Relate H(z) to FREQUENCY RESPONSE

$$H(\hat{\omega}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- NOTATION for FREQUENCY RESPONSE

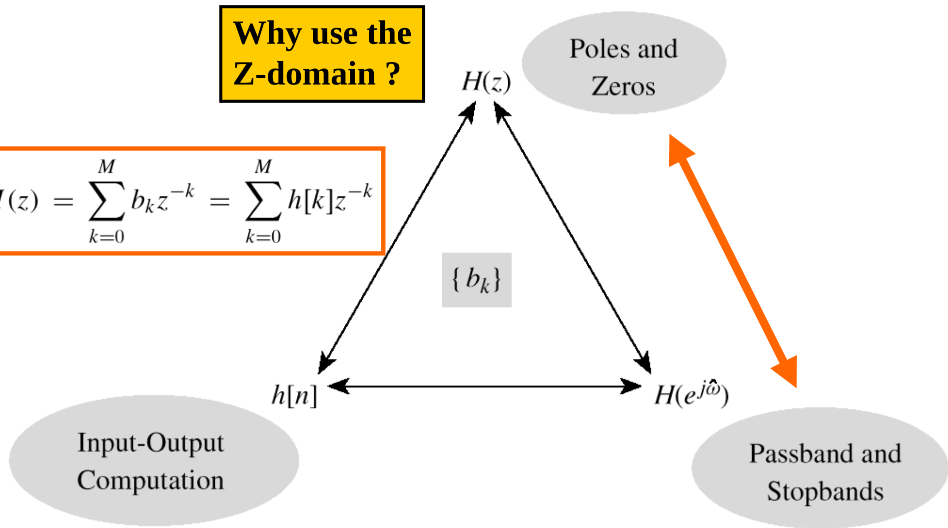
$$H(\hat{\omega}) \leftrightarrow H(e^{j\hat{\omega}})$$

ANOTHER ANALYSIS TOOL

- Why use the z–Transform ?
- The z–domain is COMPLEX
- H(z) is a COMPLEX–VALUED function of a COMPLEX VARIABLE z.

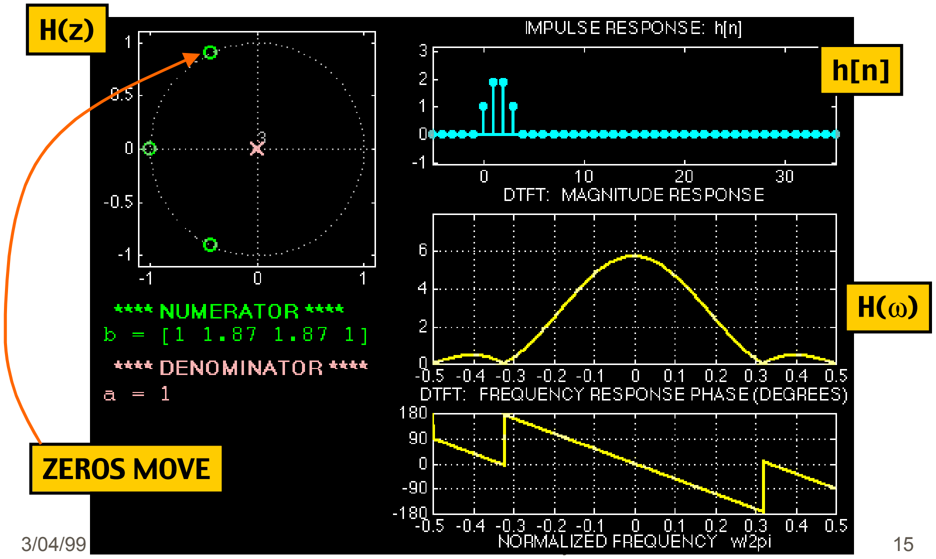
THREE DOMAINS

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \sum_{k=0}^{M} h[k] z^{-k}$$

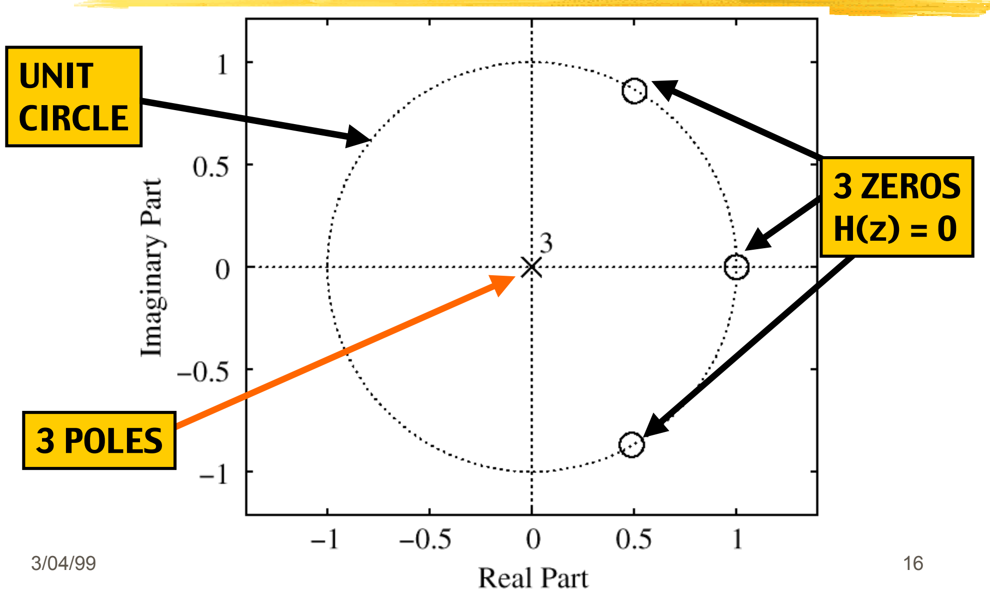


Relationship among the n-, z-, and $\hat{\omega}$-domains. The filter coefficients $\{b_k\}$ play a central role.

3 DOMAINS MOVIE: FIR

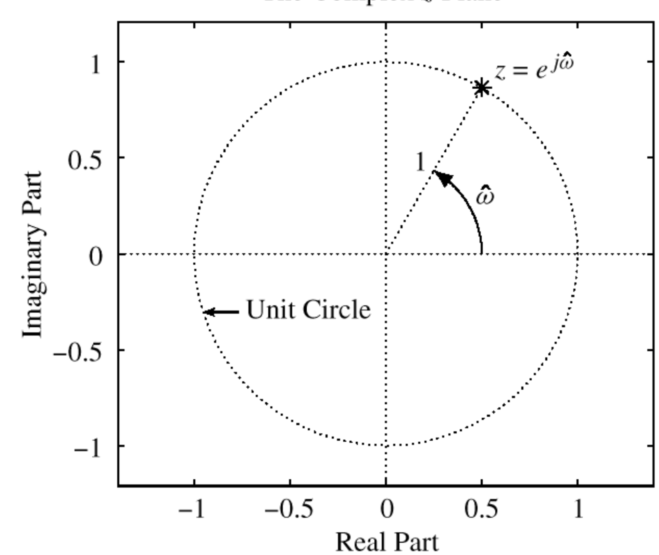


PLOT ZEROS in z-DOMAIN



$$\mathcal{H}(\hat{\omega}) = H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

The Complex z-Plane



NULLING FILTER

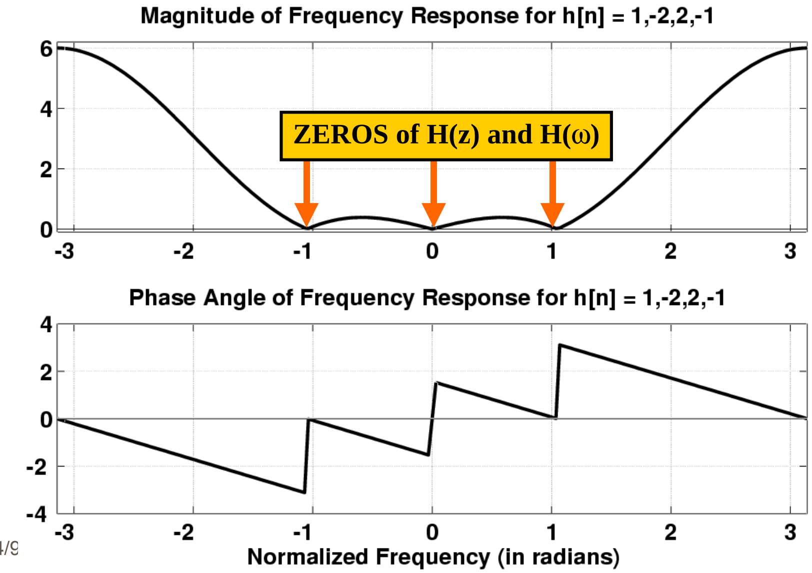
- PLACE ZEROS to make y[n] = 0

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$ ← **3 ZEROS H(z) = 0**

the output resulting from each of the following three signals will be zero:

$H(z_1) = 0$	$x_1[n] = (z_1)^n = 1$ →	$y_1[n] = 0$
$H(z_2) = 0$	$x_2[n] = (z_2)^n = e^{j\pi n/3}$	$y_2[n] = 0$
$H(z_3) = 0$	$x_3[n] = (z_3)^n = e^{-j\pi n/3}$	$y_3[n] = 0$

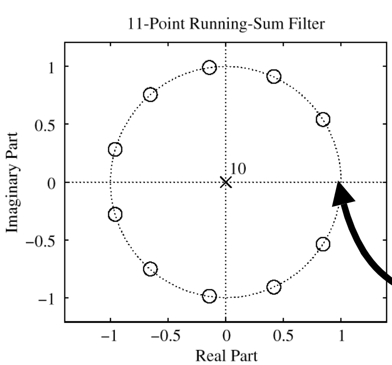
FIR Frequency Response

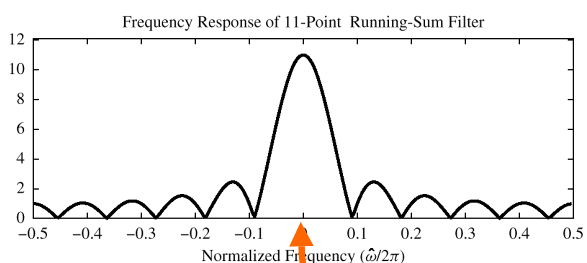


11-pt RUNNING SUM H(z)

$$H(z) = \sum_{k=0}^{10} z^{-k}$$

$$H(z) = (1 - e^{j2\pi/11}z^{-1})(1 - e^{j4\pi/11}z^{-1})\cdots(1 - e^{j20\pi/11}z^{-1})$$





NO zero at z=1

L-pt RUNNING SUM H(z)

$$H(z) = \sum_{k=0}^{L-1} z^{-k} = \frac{1 - z^{-L}}{1 - z^{-1}} = \frac{z^L - 1}{z^{L-1}(z - 1)}$$

$$z^L - 1 = 0 \implies z^L = 1$$

$$z = e^{j2\pi k/L} \quad \text{for } k = 0, 1, 2, \ldots, L-1$$

ZEROS on UNIT CIRCLE

Numerator has (z−1) term when k=0

ONE FEEDBACK TERM

- ADD **PREVIOUS** OUTPUTS

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

PREVIOUS FEEDBACK

FIR PART of the FILTER

FEED-FORWARD

- CAUSALITY
  - NOT USING **FUTURE** OUTPUTS or INPUTS

FILTER COEFFICIENTS

- ADD **PREVIOUS** OUTPUTS

$$y[n] = 0.8 y[n-1] + 3x[n] - 2x[n-1]$$

FEEDBACK COEFFICIENT

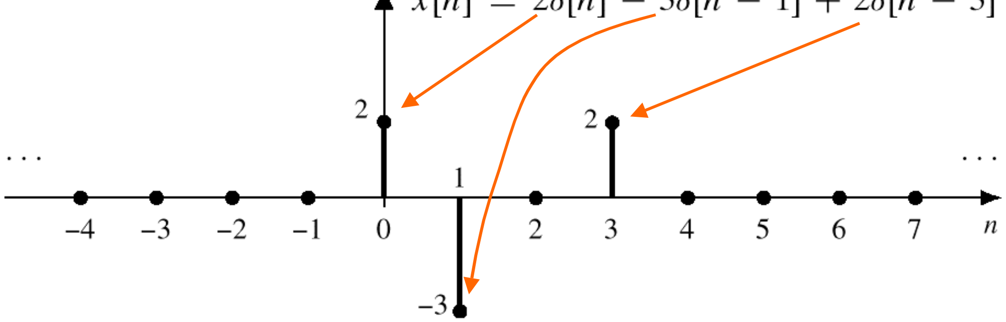
SIGN CHANGE

- MATLAB
  - `yy = filter([3,-2],[1,-0.8],xx)`

COMPUTE OUTPUT

$$y[n] = 0.8y[n-1] + 5x[n]$$

$$x[n] = 2\delta[n] - 3\delta[n-1] + 2\delta[n-3]$$



COMPUTE y[n]

- FEEDBACK DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 5x[n]$$

- NEED y[−1] to get started

$$y[0] = 0.8y[-1] + 5x[0]$$

AT REST CONDITION

- y[n] = 0, for n<0
- BECAUSE x[n] = 0, for n<0

INITIAL REST CONDITIONS

1. The input must be assumed to be zero prior to some starting time n_0, i.e., $x[n] = 0$ for $n < n_0$. We say that such inputs are *suddenly applied*.
2. The output is likewise assumed to be zero prior to the starting time of the signal, i.e., $y[n] = 0$ for $n < n_0$. We say that the system is *initially at rest* if its output is zero prior to the application of a suddenly applied input.

COMPUTE y[0]

- THIS STARTS THE RECURSION:

With the initial rest assumption, $y[n] = 0$ for $n < 0$,

$$y[0] = 0.8y[-1] + 5(2) = 0.8(0) + 5(2) = 10$$

- APPLIES TO ALL FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]$$

COMPUTE MORE y[n]

- CONTINUE THE RECURSION:

$$y[1] = 0.8y[0] + 5x[1] = 0.8(10) + 5(-3) = -7$$

$$y[2] = 0.8y[1] + 5x[2] = 0.8(-7) + 5(0) = -5.6$$

$$y[3] = 0.8y[2] + 5x[3] = 0.8(-5.6) + 5(2) = 5.52$$

$$y[4] = 0.8y[3] + 5x[4] = 0.8(5.52) + 5(0) = 4.416$$

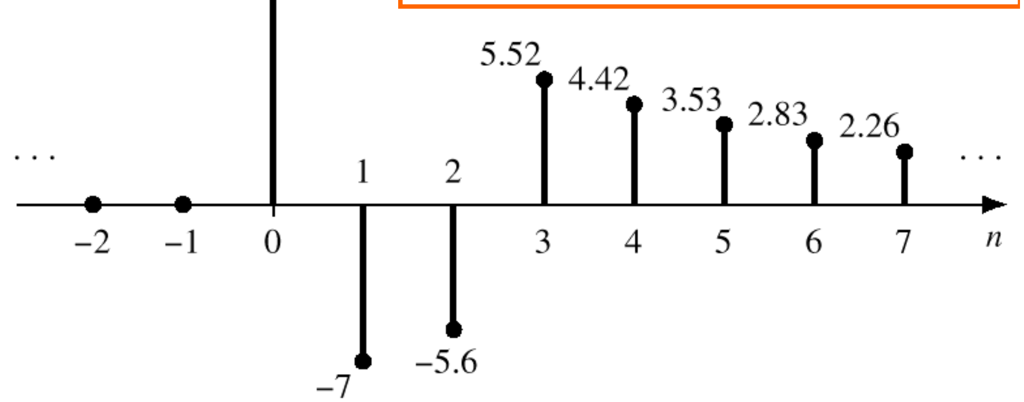
$$y[5] = 0.8y[4] + 5x[5] = 0.8(4.416) + 5(0) = 3.5328$$

$$y[6] = 0.8y[5] + 5x[6] = 0.8(3.5328) + 5(0) = 2.8262$$

PLOT y[n]

$$y[n] = 0.8y[n-1] \quad \text{for } n > 3$$

$$y[n] = y[3](0.8)^{n-3} \quad \text{for } n \geq 3$$



IMPULSE RESPONSE

n	$n<0$	0	1	2	3	4
$\delta[n]$	0	1	0	0	0	0
$h[n-1]$	0	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$
$h[n]$	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$	$b_0(a_1)^4$

From this table it is obvious that the general formula is

$$h[n] = \begin{cases} b_0(a_1)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$h[n] = b_0(a_1)^n u[n]$$

$u[n]=1, \quad \text{for } n \geq 0$

STEP RESPONSE: x[n]=u[n]

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$u[n]=1, \quad \text{for } n \geq 0$

n	$x[n]$	$y[n]$
$n<0$	0	0
0	1	b_0
1	1	$b_0 + b_0(a_1)$
2	1	$b_0 + b_0(a_1) + b_0(a_1)^2$
3	1	$b_0(1 + a_1 + a_1^2 + a_1^3)$
4	1	$b_0(1 + a_1 + a_1^2 + a_1^3 + a_1^4)$
.	.	.

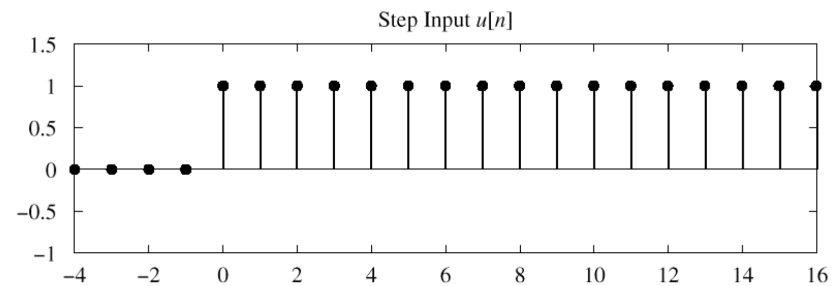
DERIVE STEP RESPONSE

$$y[n] = b_0(1 + a_1 + a_1^2 + \ldots + a_1^n) = b_0 \sum_{k=0}^{n} a_1^k$$

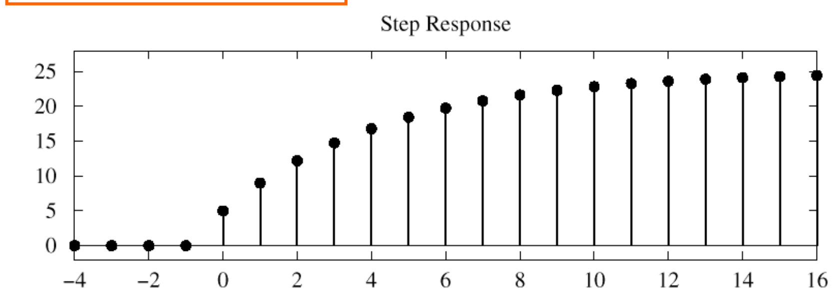
$$\sum_{k=0}^{L} r^k = \begin{cases} \dfrac{1 - r^{L+1}}{1 - r} & r \neq 1 \\ L + 1 & r = 1 \end{cases}$$

$$y[n] = b_0 \frac{1 - a_1^{n+1}}{1 - a_1} \quad \text{for } n \geq 0, \quad \text{if } a_1 \neq 1$$

PLOT STEP RESPONSE



$$y[n] = 0.8y[n-1] + 5x[n]$$



Z-Transform of IIR Filter

- **DERIVE the SYSTEM FUNCTION H(z)**
  - Use **DELAY PROPERTY**

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)$$

Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n-n_0] \iff z^{-n_0} X(z)$$

SYSTEM FUNCTION of IIR

- **NOTE the FILTER COEFFICIENTS**

$$Y(z) - a_1 z^{-1} Y(z) = b_0 X(z) + b_1 z^{-1} X(z)$$

$$(1 - a_1 z^{-1}) Y(z) = (b_0 + b_1 z^{-1}) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{B(z)}{A(z)}$$