Combined multinomial logit modal split and paired combinatorial logit traffic assignment model

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To better address the route overlap problem of the multinomial logit model used in combined modal split and traffic assignment models in the literature, this study proposes a combined multinomial logit modal split and paired combinatorial logit traffic assignment (MNL-PCL) model. The PCL model can account for the route overlap problem using a similarity index for each pair of routes in the network. It requires significantly fewer parameters to be calibrated using real-world data. Thereby, it circumvents parameter estimation issues associated with a cross-nested logit model. An equivalent mathematical programming problem is developed for the MNL-PCL model. Further, an analytical model is developed for sensitivity analysis of the MNL-PCL model. Several applications of the proposed MNL-PCL model are demonstrated using a numerical example by leveraging the results of sensitivity analysis. The study insights can assist decision-makers to design more effective strategies to promote “go-green” travel modes and reduce network congestion.

Keywords: Traffic modal split, combinatorial logit model, sensitivity analysis, “go-green” travel modes

1. Introduction

The standard “four-step” modeling of travel demand consisting of trip generation, trip distribution, modal split and traffic assignment, is often criticized by practitioners due to its inherent weaknesses such as the lack of a unified rationale to explain all aspects of demand jointly (Oppenheim 1995), inconsistencies in travel choices (Chen, Pravinvongvuth, and Chootinan 2003), error propagation across steps, etc. To address these issues, combined models, which integrate travelers’ choices across two or more steps simultaneously (e.g., trip distribution and traffic assignment), are proposed in the literature to simplify the sequential process of “four-step” travel demand models and ensure convergence to equilibrium (Lam and Huang 1992). This study proposes a combined modal split and traffic assignment (CMSTA) model. A CMSTA model provides behavioral robustness by simultaneously addressing travelers’ mode choices and mode-specific route choices. Besides, it is usually formulated as a
single mathematical programming problem, and is hence computationally tractable (Wu and Lam 2003; Kitthamkesorn and Chen 2017).

The CMSTA model has several implementation advantages in practice. First, it characterizes the impacts of modal split on network equilibrium flow. This enables decision-makers to evaluate the effectiveness of different strategies (e.g., strategies to promote “go-green” travel modes) and their impacts on network performance (e.g., level of service, total emissions, etc.). Second, solving the CMSTA model entails a ‘feedback’ mechanism which factors the impacts of the traffic assignment outcomes on the modal split model. Thereby, the modal split in the CMSTA model factors the impacts of travel time and network congestion level, unlike the standard “four-step” travel demand model. This enables decision-makers to determine how network flow changes caused by expected (e.g., planned road maintenance) or unexpected events (e.g., collapse of bridges, floods etc.) will impact travelers’ mode choice, and to estimate the demand shift between motorized travel modes and “go-green” travel modes (such as subway, bicycle, etc.).

The CMSTA model has been extensively studied in the literature. Florian (1977) developed an integrated model that simultaneously considers the demand and the equilibrium route choices on the road and transit networks. Florian and Nguyen (1978) developed an equivalent mathematical programming formulation for CMSTA which integrates a logit mode choice model and a user equilibrium route choice model. Thereafter, different models have been developed to characterize a traveler’s simultaneous mode shift and route choice behavior using variational inequality or fixed point formulations (Abdulaal and LeBlanc 1979; Leblanc and Abdulaal 1982; Shimamoto et al. 2017). The route choice model is critical for the accuracy of the CMSTA model when implementing it to forecast travel demand. Several studies seek to relax the strong behavioral assumption behind user equilibrium (UE), and use a MNL model to represent both the mode choice and route choice dimensions (labeled as
MNL-MNL model) (Oppenheim 1995; Wu and Lam 2003). However, due to the independence of irrelevant alternatives (IIA) property, the MNL model has the inherent drawback that it cannot capture the correlations among overlapped routes, and thereby overestimates flows on these routes. This leads to issues of behavioral realism associated with route choice predictions in some contexts. Recently, Kitthamkesorn et al., (2016) developed a CMSTA model to address the route overlap problem by characterizing the route choice behavior using a cross-nested logit (CNL) model. However, the CNL model contains two types of coefficients that need to be calibrated using real-world data, the similarity index coefficient, and the nest-specific coefficients whose number equals the number of links (Prato 2009). Due to their complexity, calibrating these nest-specific coefficients is computationally expensive (Prato 2009; Ramming 2001; Prato 2005). In addition, several real-world applications show that the estimated nesting coefficients often approach one, suggesting that the CNL model tends to collapse to MNL (Ramming 2001; Prato 2005, 2009). This limits the applicability of the CNL model.

To address the issues associated with the CNL model, and to better overcome the route overlap problem of the MNL route choice model in the literature, this study develops a new CMSTA model, labeled the MNL-PCL model, in which an MNL model is used for modal split and a paired combinatorial logit (PCL) model (Koppelman and Wen 2000) is used to assign the network traffic flow. The PCL model resolves the route overlap issue by introducing independent similarity relationships for each pair of routes. Further, it can be scaled to account for perception variance related to different trip lengths (Pravinvongvuth and Chen, 2005) to address the two issues in the MNL model. Also, it requires only the estimation of the parameter in the route similarity index, significantly reducing computational burden in parameter estimation. Pravinvongvuth and Chen (2005) suggest that the PCL model is more suitable for the route choice problem compared to the CNL model. It can resolve drawbacks
of the MNL model while ensuring the analytical tractability of the choice probability function. Due to these benefits, the PCL model is widely used to model travelers’ route choice behavior. Its advantages in dealing with the effects of traffic congestion, stochastic route choices, and the route similarity problem have been discussed in many studies (Chen, Kasikitwiwat, and Ji 2003; Chen et al. 2014; Pravinvongvuth and Chen 2005; Chen et al. 2012). Recently, Ryu et al. (2014) extend the PCL model to predict the demand pattern by considering demand elasticity. Karoonsoontawong and Lin (2015) developed a combined trip distribution and traffic assignment model by integrating the gravity model into the PCL model to simultaneously determine trip distribution and assignment for each origin-destination (O-D) pair. Wen and Koppelman (2001) applied the PCL model to estimate traveler’s traffic mode choices to account for the correlations between different traffic modes.

This study develops an equivalent mathematical programming problem for the MNL-PCL model. It then conducts sensitivity analysis of the MNL-PCL model. Further, the study explores several applications based on the results of the sensitivity analysis, including: (1) estimating the perturbed solution of the MNL-PCL model to provide an estimate of the traffic flow and mode-specific demand after network parameters (e.g., total demand, link capacity, etc.) are perturbed by expected or unexpected events, (2) performing critical parametric analysis to identify the parameters whose control would enhance system performance (e.g., reduce emissions, increase “go-green” travel mode demand, etc.), (3) analyzing impacts of traffic control measures (e.g., access control) on road network equilibrium and travelers’ mode choice, and (4) performing uncertainty analysis to provide insights on confidence levels for mode-specific demand and network flow when perturbed parameters are uncertain. The proposed MNL-PCL model and analytical sensitivity analysis can assist planners and operational decision-makers to design effective policies and control strategies to promote “go-green” travel modes, alleviate traffic congestion, and reduce green-house gas emissions.
The remainder of the paper is organized as follows. In the next section, an equivalent mathematical programming problem is developed for the MNL-PCL model. Then, an analytical formulation is proposed for sensitivity analysis of the MNL-PCL model. Thereafter, several applications of the MNL-PCL model and corresponding sensitivity analyses are illustrated using numerical examples. The final section provides some concluding comments and insights.

2. Combined multinomial logit modal split and paired combinatorial logit traffic assignment model

2.1 Paired combinatorial logit model

The PCL model was initially derived by Chu (1989) for travel demand analysis. It was adopted by Prashker and Bekhor (1999) to model route choice to resolve the route overlap problem. It overcomes the independence assumption associated with MNL model by allowing different covariances between the utilities of pairs of alternatives. For the routes between an arbitrary O-D pair \( r - s \) in the network of mode \( m \) (e.g., auto mode network), the PCL model has a hierarchical structure that decomposes the choice probability into two levels represented by the marginal probabilities \( P_m^{rs}(k) \) in the upper level and conditional probabilities \( P_m^{rs}(k | kj) \) in the lower level (see Figure 1); here \( k \) and \( j \) are two route alternatives between the O-D pair \( r - s \) in the network of mode \( m \). The PCL choice probability can be expressed as:

\[
P_m^{rs}(k) = \sum_{j \neq k} P_m^{rs}(kj) \cdot P_m^{rs}(k | kj)
\]

where

\[
P_m^{rs}(k | kj) = \frac{\theta_{c_{m}^{rs}}^{kj}}{e^{\sigma_{c_{m}^{rs}}^{kj}} + e^{1 - \sigma_{c_{m}^{rs}}^{kj}}} \cdot \frac{\theta_{e_{m}^{rs}}^{kj}}{e^{\sigma_{e_{m}^{rs}}^{kj}} + e^{1 - \sigma_{e_{m}^{rs}}^{kj}}},
\]

(1)
\[ P_m^{rs}(kj) = \frac{(1 - \sigma_{m,kj}^{rs}) \left( \frac{\theta_{m,kj}^{rs}}{e^{1-\sigma_{m,kj}^{rs}}} + \frac{\theta_{m,jk}^{rs}}{e^{1-\sigma_{m,jk}^{rs}}} \right)^{1-\sigma_{m,kj}^{rs}}}{\sum_{l=1}^{n-1} \sum_{h=l+1}^{n} (1 - \sigma_{m,hl}^{rs}) \left( \frac{\theta_{m,lh}^{rs}}{e^{1-\sigma_{m,lh}^{rs}}} + \frac{\theta_{m,hk}^{rs}}{e^{1-\sigma_{m,hk}^{rs}}} \right)^{1-\sigma_{m,lh}^{rs}}} \] (3)

where \( \theta_n \) is the dispersion parameter for network of mode \( m \); \( c_{m,k}^{rs} \) is the travel time of route \( k \), and \( n \) is the number of alternative routes between O-D pair \( r - s \) in the network of mode \( m \).

\( P_m^{rs}(k \mid kj) \) denotes the conditional probability of choosing route \( k \) given that the alternative route pair \( kj \) has been chosen, and \( P_m^{rs}(kj) \) is the marginal (unobserved) probability for the alternative route pair \( kj \). \( \sigma_{m,kj}^{rs} \) is the similarity index between alternatives \( k \) and \( j \) connecting O-D pair \( r - s \) in the network of mode \( m \). It is formulated as a function of the shared links between routes (Chu, 1989):

\[ \sigma_{m,kj}^{rs} = \left( \frac{L_{kj}}{\sqrt{L_k \sqrt{L_j}}} \right)^\zeta \] (4)

where \( L_{kj} \) is the length of the common part of routes \( k \) and \( j \), \( L_k \) and \( L_j \) are the lengths of routes \( k \) and \( j \), respectively, and \( \zeta \) is a parameter to be estimated. As can be seen, the PCL model only involves parameter \( \zeta \) in the similarity index function that needs to be calibrated.

By comparison, the CNL model requires the calibration of not only the parameter in the similarity index function, but also the nest-specific coefficients (Prato 2005). Hence, the PCL model has significantly lower computational complexity for parameter estimation compared to the CNL model.

Figure 1 provides an illustration of the hierarchical tree structure of the PCL model for a simple network with three alternative routes between origin node \( O \) and destination node \( D \). The route overlap problem is highlighted by introducing the similarity index terms for each
pair of routes. As seen in the figure, the similarity indices (i.e., $\sigma_{12}$ and $\sigma_{13}$) between route 1 and the other two routes are 0 as there are no overlapping links between them. The similarity index ($\sigma_{23}$) between routes 2 and 3 is positive, and is proportional to the length $L_B$ of their overlapping link B.

2.2 Mathematical formulation of the MNL-PCL model

To overcome the route overlap problem associated with the MNL-based combined model in the literature (Oppenheim 1995), an equivalent mathematical programming formulation for the MNL-PCL model is developed in this section, where the MNL model is used to capture the travelers’ travel mode choice behavior and the PCL model is used to characterize the travelers’ route choice behavior. The following two assumptions are made to develop the mathematical formulation for MNL-PCL model.

Assumption 1 The network of each travel mode is independent of other modal networks. That is, modal networks do not share common links.

Assumption 2 The link travel time is separable, and is a monotonic non-decreasing function of traffic flows.

In the MNL-PCL model, the marginal mode choice probability is:

$$
\frac{q_{rs}^{\text{m}}}{\sum_{g \in M_{rs}} q_{gs}^{\text{m}}} = \frac{\exp(\gamma_{rs}^{\text{m}} (\Psi_{rs}^{\text{m}} - w_{rs}^{\text{m}}))}{\sum_{g \in M_{rs}} \exp(\gamma_{rs}^{\text{m}} (\Psi_{gs}^{\text{m}} - w_{gs}^{\text{m}}))}
$$

where $q_{rs}^{\text{m}}$ is the O-D demand for O-D pair $r$-$s$ of mode $m$, $\gamma_{rs}^{\text{m}}$ is the O-D specific parameter, and $w_{rs}^{\text{m}}$ is the expected perceived travel (EPT) time for O-D pair $r$-$s$ of mode $m$. $\Psi_{rs}^{\text{m}}$ denotes the exogenous attractiveness of O-D pair $r$-$s$ of mode $m$; for example, it can be due to lower travel expenses, lower carbon footprint, etc. (Kitthamkesorn et al. 2016). $M_{rs}$ is the set of travel modes.
For each mode, the route choice probability for O-D pair \( r-s \) in the MNL-PCL model complies with the PCL model Eqs. (1)-(3). Denote \( \sigma_{m,kj}^{rs} \) as the similarity index between routes \( k \) and \( j \) connecting origin \( r \) and destination \( s \) using mode \( m \). Let \( \beta_{m,kj}^{rs} = 1 - \sigma_{m,kj}^{rs} \). The MNL-PCL model is formulated as:

\[
\min H = Z_1 + Z_2 + Z_3 + Z_4 + Z_5
\]

\[
Z_1 = \sum_{m \in M} \sum_{a \in A_m} \int_0^{v_m} t_{m,a} (w) dw,
\]

\[
Z_2 = \sum_{rs} \sum_{m \in M_{rs}} \frac{1}{\theta_m} \sum_{k \neq k} \beta_{m,kj}^{rs} f_{m,k(j)}^{rs} \ln \left( \frac{f_{m,k(j)}^{rs}}{\beta_{m,kj}^{rs}} \right),
\]

\[
Z_3 = \sum_{rs} \sum_{m \in M_{rs}} \frac{1}{\theta_m} \sum_{k=1}^{k^{rs}+1} \sum_{j=k+1} f_{m,k(j)}^{rs} (1 - \beta_{m,kj}^{rs}) (f_{m,k(j)}^{rs} + f_{m,j(j)}^{rs}) \ln \left( \frac{f_{m,k(j)}^{rs} + f_{m,j(j)}^{rs}}{\beta_{m,kj}^{rs}} \right),
\]

\[
Z_4 = \sum_{rs} \sum_{m \in M_{rs}} \left( \frac{1}{\gamma^{rs} - \theta_m} - \frac{1}{\theta_m} \right) q_m^{rs} \ln q_m^{rs} - \frac{1}{\gamma^{rs}} q_m^{rs},
\]

\[
Z_5 = -\sum_{rs} \sum_{m \in M_{rs}} q_m^{rs} \psi_m^{rs},
\]

s.t. \( \sum_{k \neq k} f_{m,k(j)}^{rs} = q_m^{rs}, \forall r, s; \forall m \in M_{rs}, \)

\[
\sum_{m \in M_{rs}} q_m^{rs} = q^{rs}, \forall r, s,
\]

\[
f_{m,k(j)}^{rs} \geq 0, \forall r, s, m, k, j
\]

where \( A_m \) is the set of links on the network of mode \( m \), \( v_{m,a} \) is the flow on link \( a \) using mode \( m \), \( f_{m,k(j)}^{rs} \) is the flow on route \( k \) (of the route pair \( kj \)) of O-D pair \( r-s \) using mode \( m \), \( K_m^{rs} \) is the set of all routes on the network of mode \( m \) for O-D pair \( r-s \), and \( q_m^{rs} \) and \( q^{rs} \) are the travel demand of mode \( m \) and the total demand of O-D pair \( r-s \), respectively.

The objective function of MNL-PCL (Eq. (6)) consists of five terms: \( Z_1 \) is the user
equilibrium (UE) term reflecting the congestion effect; $Z_2$ and $Z_3$ are entropy terms reflecting the stochasticity effect and the similarity effect, respectively. $Z_4$ and $Z_5$ respectively handle the conditional probability of MNL modal split and the given exogenous modal attractiveness. Eq. (12) ensures that for a specific mode, the flow on all routes for an O-D pair sums up to the O-D demand of that mode. Eq. (13) is the demand conservation constraint, and Eq. (14) ensures the non-negativity of flow.

**Proposition 1** Solving the mathematical programming formulation in Eqs. (6)-(14) provides the route choice solution of the PCL model and the mode choice solution of the MNL model.

**Proof:** The equivalency can be shown by deriving the Karush-Kuhn-Tucker (KKT) conditions for optimality with respect to route flows ($f_{m,k(j)}^{rs}$, $\forall r,s,m,k,j$) and O-D demands ($q_m^{rs}$, $\forall r,s,m$), respectively. The Lagrangian equation for the MNL-PCL model is given by:

$$L = H + \sum_{m} \sum_{k} \sum_{j} \mu_m^{rs} \left( f_{m,k(j)}^{rs} - q_m^{rs} \right) + \sum_{m} \sum_{k} \tau^{rs} \left( q_m^{rs} - q^{rs} \right)$$

(15)

where $\mu_m^{rs}$ and $\tau^{rs}$ are Lagrange multipliers associated with Eq. (12) and Eq. (13), respectively. Under assumption 1, the network of each mode is independent. Then, for each mode $m$, the KKT conditions at the equilibrium solution can be written as:

$$\nabla_{f_{m,k(j)}^{rs}} L = c_{m,k}^{rs} + \frac{\beta_{m,k}^{rs}}{\theta_m} \ln \left( \frac{f_{m,k(j)}^{rs}}{\beta_{m,k}^{rs}} \right) + \frac{1}{\theta_m} \ln \left( \frac{f_{m,k(j)}^{rs} + f_{m,j(k)}^{rs}}{\beta_{m,k}^{rs}} \right) + \frac{1}{\theta_m} + \mu_m^{rs} = 0, \forall r,s,k$$

(16)

$$\nabla_{q_m^{rs}} L = \left( \frac{1}{\gamma^{rs}} - \frac{1}{\theta_m} \right) \ln q_m^{rs} - \frac{1}{\theta_m} - \Psi_m^{rs} - \mu_m^{rs} + \tau^{rs} = 0, \forall r,s.$$ 

(17)

$$f_{m,k(j)}^{rs} \geq 0, \forall r,s,k,j; \quad \mu_m^{rs} \geq 0, \forall r,s; \quad \tau^{rs} \geq 0, \forall r,s.$$ 

The proof of proposition 1 can be divided into the following two steps.

**Step 1:** The route flow solution of the formulation (Eqs. (6)-(14)) coincides with the PCL model.
Proof: Eq. (16) can be rewritten as:

\[
\frac{f_{rs}^{m,k(k)}}{\beta_{m,kj}^{rs}} \left( \frac{f_{rs}^{m,j(k)}}{f_{m,j(k)}} \right) \exp \left( \frac{\theta_m c_{m,k}^{rs}}{\beta_{m,kj}^{rs}} - \frac{1}{\beta_{m,kj}^{rs}} - \frac{\theta_m c_{m,j}^{rs}}{\beta_{m,kj}^{rs}} \right) = \exp \left( \frac{\theta_m c_{m,k}^{rs}}{\beta_{m,kj}^{rs}} - \frac{1}{\beta_{m,kj}^{rs}} - \frac{\theta_m c_{m,j}^{rs}}{\beta_{m,kj}^{rs}} \right) .
\]  

Similarly, for a route \( j \) different from \( k \), the following equation holds:

\[
\frac{f_{rs}^{m,j(k)}}{\beta_{m,kj}^{rs}} \left( \frac{f_{rs}^{m,j(k)}}{f_{m,j(k)}} \right) \exp \left( \frac{\theta_m c_{m,j}^{rs}}{\beta_{m,kj}^{rs}} - \frac{1}{\beta_{m,kj}^{rs}} - \frac{\theta_m c_{m,j}^{rs}}{\beta_{m,kj}^{rs}} \right) = \exp \left( \frac{\theta_m c_{m,k}^{rs}}{\beta_{m,kj}^{rs}} - \frac{1}{\beta_{m,kj}^{rs}} - \frac{\theta_m c_{m,j}^{rs}}{\beta_{m,kj}^{rs}} \right) .
\]  

Dividing Eq. (18) by Eq. (19), yields

\[
\frac{f_{rs}^{m,k(k)}}{f_{m,j(k)}} \exp \left( \frac{\theta_m c_{m,k}^{rs}}{\beta_{m,kj}^{rs}} - \frac{1}{\beta_{m,kj}^{rs}} - \frac{\theta_m c_{m,j}^{rs}}{\beta_{m,kj}^{rs}} \right) = \exp \left( \frac{\theta_m c_{m,k}^{rs}}{\beta_{m,kj}^{rs}} - \frac{1}{\beta_{m,kj}^{rs}} - \frac{\theta_m c_{m,j}^{rs}}{\beta_{m,kj}^{rs}} \right) .
\]

Eq. (20) gives conditional probability of choosing route \( k \) from route pair \( k,j \):

\[
\frac{f_{rs}^{m,k(k)}}{f_{m,j(k)}} = \exp \left( \frac{\theta_m c_{m,k}^{rs}}{\beta_{m,kj}^{rs}} - \frac{1}{\beta_{m,kj}^{rs}} - \frac{\theta_m c_{m,j}^{rs}}{\beta_{m,kj}^{rs}} \right) \exp \left( \frac{\theta_m c_{m,j}^{rs}}{\beta_{m,kj}^{rs}} - \frac{1}{\beta_{m,kj}^{rs}} - \frac{\theta_m c_{m,j}^{rs}}{\beta_{m,kj}^{rs}} \right) .
\]

According to Eq. (18) and Eq. (19), we have

\[
f_{rs}^{m,k(k)} + f_{m,j(k)} = \beta_{m,kj}^{rs} \exp(-\theta_m \mu_{m}^{rs}) \left[ \exp \left( \frac{\theta_m c_{m,k}^{rs}}{\beta_{m,kj}^{rs}} - \frac{1}{\beta_{m,kj}^{rs}} - \frac{\theta_m c_{m,j}^{rs}}{\beta_{m,kj}^{rs}} \right) \right] .
\]

Eq. (12) indicates that

\[
q_m^{rs} = \sum_{l=1}^{K_l} \sum_{h=1}^{K_h} \left( f_{rs}^{m,l(h)} + f_{m,h(l)} \right)
\]

\[
= \exp(-\theta_m \mu_{m}^{rs}) \left[ \sum_{l=1}^{K_l} \sum_{h=1}^{K_h} \beta_{m,lh}^{rs} \exp \left( \frac{\theta_m c_{m,lh}^{rs}}{\beta_{m,lh}^{rs}} \right) \right] .
\]

The marginal condition probability of the PCL model of choosing pair \( k,j \) is obtained by dividing Eq. (22) by Eq. (23), as follows:
\[
\frac{f_{m,i(j)}^{rs} + f_{m,j(k)}^{rs}}{\sum_{l=1}^{k_{m,i}^{rs}} \sum_{h=1+1}^{k_{m,j}^{rs}} (f_{m,(i(h))}^{rs} + f_{m,(h(j))}^{rs})} = \beta_{m,ij}^{rs} \left[ \exp \left( -\frac{\theta_m c_{m,k}^{rs}}{\beta_{m,ij}^{rs}} \right) + \exp \left( -\frac{\theta_m c_{m,l}^{rs}}{\beta_{m,ij}^{rs}} \right) \right]^{-1}.
\] (24)

Eq. (21) and Eq. (24) denote that both the conditional probability (Eq. (2)) and marginal condition probability (Eq. (3)) of PCL model are satisfied. Thereby, Step 1 is proved.

**Step 2**: The modal split solution of formulation (Eqs. (6)-(14)) coincide with the MNL model.

**Proof**: According to Eq. (23), we have

\[
\mu_m^{rs} = -\frac{1}{\theta_m} \ln \left( q_m^{rs} \right) - \frac{1}{\theta_m} \ln \left( \sum_{k=1}^{k_{m}^{rs}} \sum_{j=1}^{k_{m}^{rs}} \beta_{m,ij}^{rs} \left[ \exp \left( -\frac{\theta_m c_{m,k}^{rs}}{\beta_{m,ij}^{rs}} \right) + \exp \left( -\frac{\theta_m c_{m,l}^{rs}}{\beta_{m,ij}^{rs}} \right) \right]^{-1} \right).
\] (25)

Ryu et al. (2014) shows that for mode \( m \), the EPT time of an O-D pair \( r - s \) can be written as:

\[
w_m^{rs} = -\frac{1}{\theta_m} \ln \left( \sum_{k=1}^{k_{m}^{rs}} \sum_{j=1}^{k_{m}^{rs}} \beta_{m,ij}^{rs} \left[ \exp \left( -\frac{\theta_m c_{m,k}^{rs}}{\beta_{m,ij}^{rs}} \right) + \exp \left( -\frac{\theta_m c_{m,l}^{rs}}{\beta_{m,ij}^{rs}} \right) \right]^{-1} \right).
\] (26)

Substituting Eq.(25) and Eq. (26) into Eq.(17), yields,

\[
\left( \frac{1}{\gamma^{rs}} - \frac{1}{\theta_m} \right) \ln q_m^{rs} - \frac{1}{\theta_m} - \Psi_m^{rs} + \frac{1}{\theta_m} \ln \left( q_m^{rs} \right) + \frac{1}{\theta_m} + w_m^{rs} = 0.
\] (27)

Thereby, we have

\[
q_m^{rs} = \exp \left( \gamma^{rs} (\Psi_m^{rs} - w_m^{rs}) - \gamma^{rs} \tau^{rs} \right).
\] (28)

Eq. (28) indicates that the conditional probability of choosing mode \( m \) to travel O-D pair \( r - s \) is:

\[
\sum_{g \in M_{rs}} q_g^{rs} = \frac{\exp\left( \gamma^{rs} (\Psi_m^{rs} - w_m^{rs}) \right)}{\exp\left( \gamma^{rs} (\Psi_g^{rs} - w_g^{rs}) \right)}.
\] (29)

Eq. (29) is consistent with the MNL choice probability model (Eq.(5)). Step 2 is proved.
According to Steps 1 and 2, proposition 1 holds. This completes the proof.

To illustrate the uniqueness of the solution of MNL-PCL model, the following assumption is used in this study.

**Assumption 3** The values of all O-D specific parameters $\gamma^r$, $\forall r, s$ are less than the values of the dispersion parameter of any mode, that is, $\gamma^r < \theta_m$, $\forall r, s, m$.

Note that the variance of the EPT time for an O-D pair $r-s$ and the variances of the perceived travel times of paths in the network of mode $m$ are inversely proportional to $\gamma^r$ and $\theta_m$, respectively. Assumption 3 holds when the variance of the perceived mode-specific EPT time of each O-D pair is significantly larger than the variances of the perceived travel time for all paths.

**Proposition 2** The solution of the MNL-PCL model is unique.

**Proof**: The uniqueness of the solution of MNL-PCL model can be shown by demonstrating the strict convexity of the equivalent nonlinear optimization programming problem (Eqs. (6)-(14)).

Let $\bar{f}$ be the vector of route flows, $\bar{q} = \left[f_{m,k(ij)}, \forall r, s, m, k, j \right]$, and $q$ be the vector of O-D demands for all modes, $q = \left[q^r_m, \forall r, s, m \right]$. The Hessian matrix of the objective function (i.e., Eq. (6)) with respect to $\bar{f}$ and $q$ is:

$$
H_f = \nabla^2_{(\bar{f},q)} H = \begin{bmatrix}
\nabla^2_{\bar{f}}(Z_1 + Z_2 + Z_3) \\
\nabla^2_{q}(Z_4 + Z_5)
\end{bmatrix}
$$

(30)

where

$$
\nabla^2_{\bar{f}} Z_1 = \left[ \frac{\partial c^r_{m,k}}{\partial f_{m,k(ij)}}, \forall r, s, m, k, j \right],
$$

(31)

$$
\nabla^2_{q} Z_2 = \text{diag} \left[ \frac{\beta_{kj}}{\theta_{m} f_{m,k(ij)}}, \forall r, s, m, k, j \right],
$$

(32)
Prashker and Bekhor (1999) showed that $\nabla^2 Z_1$ is positive semidefinite, and $\nabla^2 Z_2$ and $\nabla^2 Z_3$ are positive definite under assumption 2. Thereby, $\nabla^2 (Z_1 + Z_2 + Z_3)$ is positive definite. Note that under assumption 3, $\gamma^{rs} < \theta_m$, $\forall r,s,m$, $\nabla^2 L$ is also a positive definite matrix. This implies that the matrix $\nabla^2 (L_{aq})$ is positive definite. As the constraints in Eqs. (12)-(14) are linear, the equivalent optimization problem for MNL-PCL model has a unique solution. This completes the proof.

To solve the proposed MNL-PCL model, this study proposes a route-based partial linearization method. It is used extensively for solving combined traffic assignment models (Ryu et al. 2014). Denote $|M_{rs}|$ as the total number of modes available for O-D pair $r-s$, and $|K_{rs,m}|$ as the total number of routes for O-D pair $r-s$ in the network of mode $m$. The detailed steps to implement this method for solving the MNL-PCL model are as follows:

Step 1: Initialization. Set $n = 1$; assign the total travel demand equally to all modes for each O-D pair, i.e., $q^{rs(n)} = \left(1/|M_{rs}|\right) q^{rs}$. Then, assign the mode-specific demand equally to all paths of the corresponding O-D pairs, i.e., $f^{rs(n)}_{m,k(j)} = \left(1/|K_{rs,m}|\right) q^{rs}_m$, $\forall k, j \in K_{rs,m}^r$, $\forall m$.

Step 2: Find the descent direction. Update the link travel time and the route travel time $c^{rs}_{m,k}$, $\forall r,s,m,k$ based on $f^{rs(n)}_{m,k(j)}$, $\forall r,s,m,k,j$. For each mode, perform the PCL loading between each O-D pair using the O-D demand $q^{rs(n)}_m$ according to Eqs. (1)-(3); denote the auxiliary route flow as $\tilde{f}^{rs(n)}_{m,k(j)}$, $\forall r,s,m,k,j$. Compute the expected perceived O-D
travel time for each mode (e.g., \( w_m^{rs} \)) according to Eq. (26). Obtain the auxiliary O-D demands for each mode using Eq. (5) and \( q_{rs}^{(n)} \). Denote the auxiliary O-D demand for mode \( m \) as \( \tilde{q}_m^{rs(n)} \).

Step 3: Step size. Use the self-regulated averaging method (Liu, He, and He 2009) to find the step size \( \alpha_1^{n} \) and \( \alpha_2^{n} \) for route flows and mode-specific O-D demand, respectively.

Step 4: Flow and mode-specific O-D demand update.

\[
f_{m,k(j)}^{rs(n+1)} = (1 - \alpha_1^{n}) f_{m,k(j)}^{rs(n)} + \alpha_1^{n} \tilde{f}_{m,k(j)}^{rs(n)};
\]

\[
q_{m}^{rs(n+1)} = (1 - \alpha_2^{n}) q_{m}^{rs(n)} + \alpha_2^{n} \tilde{q}_{m}^{rs(n)}.
\]

Step 5: Convergence test. If

\[
\sqrt{\sum_{m \in M} \sum_{rs} \sum_{k} \sum_{j \in k} (f_{m,k(j)}^{rs(n)} - f_{m,k(j)}^{rs(n+1)})^2} < \varepsilon_1, \quad \text{and} \quad \sqrt{\sum_{rs} \sum_{m} (q_{m}^{rs(n+1)} - q_{m}^{rs(n)})^2} < \varepsilon_2,
\]

where \( \varepsilon_1 \) and \( \varepsilon_2 \) are predetermined thresholds, then end iteration. Otherwise, set \( n = n + 1 \), go to step 2.

3. Sensitivity analysis of the MNL-PCL model

While the MNL-PCL model overcomes the route overlap issue of the MNL-MNL model, this advantage is obtained by trading off computational efficiency. Note that because the number of paired routes in the PCL model increases rapidly with network size, it is computationally expensive to solve for the MNL-PCL model for a large-sized network under parametric perturbation. Further, when expected or unexpected events occur, it is often difficult for planners and decision-makers to design effective control strategies to reduce their negative effects, and to evaluate their impacts on network performance (e.g., congestion, modal split of “go-green” modes). To address these three issues, an analytical model is proposed for sensitivity analysis of the MNL-PCL model in the next section. The sensitivity analysis determines the change in the objective function value for a unit change in the value of an
explanatory variable. It has multiple applications in practice, including estimation of the perturbed solution, and identification of critical parameters to design more effective control strategies to improve network performance.

In the literature, analytical formulations have been extensively explored for sensitivity analysis of traffic assignment models, including the UE problem (Tobin and Friesz 1988; Wang, He, and Peeta 2016), UE with elastic demand problem (Yang 1997), and stochastic user equilibrium problems (Clark and Watling 2000; Clark and Watling 2002; Ying and Miyagi 2001). Next, the analytical model for sensitivity analysis of the MNL-PCL model will be developed based on Fiacco (Fiacco 1983).

3.1 Sensitivity analysis for a general nonlinear programming formulation

Consider the nonlinear programming formulation:

$$
\min_{x} \quad z(x, \varepsilon) \tag{35}
$$

s.t.

$$
g_{i}(x, \varepsilon) \geq 0 \quad (i = 1, \ldots, \chi_{1}) \tag{36}
$$

$$
h_{j}(x, \varepsilon) = 0 \quad (j = 1, \ldots, \chi_{2}) \tag{37}
$$

where $x$ is a vector of decision variables and $\varepsilon$ is a perturbed parameter that leads to small changes in the objective function or the constraints. The nonlinear programming formulation can be reformulated as the equivalent Lagrangian expression:

$$
L_{1}(x, v, \omega, \varepsilon) = z(x, \varepsilon) - \sum_{i} v_{i} g_{i}(x, \varepsilon) + \sum_{j} \omega_{j} h_{j}(x, \varepsilon) \tag{38}
$$

where $v$ and $\omega$ are the vectors of Lagrange multipliers associated with the inequality constraints $g_{i}(x, \varepsilon), i = 1, \ldots, \chi_{1}$ and equality constraints $h_{j}(x, \varepsilon), j = 1, \ldots, \chi_{2}$, respectively.

Fiacco (1983) shows that if the nonlinear program satisfies the following four conditions: (1) the nonlinear programming problem is twice continuously differentiable in some neighborhood of $(x^*, 0)$, (2) the second-order sufficient conditions for a strict local minimum hold at $x^*$, (3) strict complementary slackness holds at $(x^*, 0)$, i.e. $v_{i}^* > 0$ when $g_{i}(x^*, 0) = 0$, (4) the

and (4) the gradients $\nabla_{x_i} g_i(x^*, 0)$ for all $i$ (where $g_i(x^*, 0) = 0$) and $\nabla_{x_j} h_j(x^*, 0)$ for all $j$ are linearly independent, then the following equations hold:

$$
\nabla_{x_i} L_i(x^*(\varepsilon), v^*(\varepsilon), \omega^*(\varepsilon), \varepsilon) = 0
$$

$$
v^*_i(\varepsilon) g_i(x^*(\varepsilon), \varepsilon) = 0, \quad i = 1, \ldots, \chi_1
$$

$$
h_j(x^*(\varepsilon), \varepsilon) = 0 \quad j = 1, \ldots, \chi_2
$$

Denote the Jacobians of the above equations with respect to solutions $y_i = [x, v, \omega]^T$ by $M_i(\varepsilon)$ and with respect to $\varepsilon$ by $N_i(\varepsilon)$. Then, we have

$$
\nabla_{\varepsilon} y_i(\varepsilon) = -M_i(\varepsilon)^{-1}N_i(\varepsilon)
$$

(39)

3.2 Sensitivity analysis for the MNL-PCL model

Here, we will illustrate that the four conditions for the sensitivity analysis of Eqs. ((35)-38)) will be satisfied by the MNL-PCL model (Eqs. (6)-(14)). Note that the decision variables in the MNL-PCL model include the route flow and the mode-specific O-D demand; thereby, $x = (\bar{f}, q)$. According to assumption 2, the objective function of MNL-PCL model (i.e., Eq. (6)) is twice continuously differentiable with respect to $(\bar{f}, q)$, and the constraint functions are also twice continuously differentiable with respect to $(\bar{f}, q)$. Thereby, condition (1) is satisfied. Proposition 2 shows that $\nabla^2_{(\bar{f}, q)} H > 0$ under assumption 3. This indicates that condition (2) is satisfied. Note that as the route choice probabilities in Eqs. (1)-(3) are strictly positive, all route flows are positive, and the inequality constraint (Eq. (14)) can be removed with no impact on the solution. Thereby, strict complementary slackness holds at $\varepsilon = 0$, and condition (3) is satisfied. Let $\Lambda = [a'_{m,k}, \forall r, s, m, k]$ be the incidence matrix for routes in vector $\bar{f}$, where $a'_{m,k} = 1$, if the $k$th route in vector $\bar{f}$ connects origin node $r$ and destination node $s$ through mode $m$; $a'_{m,k} = 0$ otherwise. Let $\Phi = [b'_{m}, \forall r, s, m]$ be the mode-OD incidence
matrix, where $b_{rs}^m = 1$ if mode $m$ is available for a traveler to travel from origin node $r$ and destination node $s$; $b_{rs}^m = 0$ otherwise. The first-order derivatives of Eq. (12) and Eq. (13) with respect to $(\mathbf{f}, \mathbf{q})$ are:

$$
\begin{bmatrix}
\Lambda & -E_{n \times n} \\
0 & \Phi
\end{bmatrix}
$$

where $n$ is the number of entries in vector $\mathbf{q}$. $E_{n \times n}$ is an identity matrix. By definition, the matrices $\Lambda$ and $\Phi$ are linearly independent. Thereby, the matrix $\begin{bmatrix}
\Lambda & -E_{n \times n} \\
0 & \Phi
\end{bmatrix}$ is linearly independent. Condition (4) is also satisfied. This indicates that Eq. (39) can be used to analyze the sensitivity of the solutions of the MNL-PCL model with respect to the perturbed parameters (for example, link capacity, signal split, link toll, etc.).

Let $y_2 = (f, q, \mu, \tau)$. Then, the first-order derivative of the KKT conditions of the MNL-PCL model (Eq. (16) and Eq. (17)) with respect to perturbed parameters can be written as:

$$
\nabla_\varepsilon y_2(\varepsilon) = -M_2(\varepsilon)^{-1}N_2(\varepsilon)
$$

where

$$
M_2(\varepsilon) = \begin{bmatrix}
\nabla_f^2(Z_1 + Z_2 + Z_3) & 0 & \Lambda^T & 0 \\
0 & \nabla_q^2(Z_4 + Z_5) & -E_{n \times n} & \Phi^T \\
\Lambda & -E_{n \times n} & 0 & 0 \\
0 & \Phi & 0 & 0
\end{bmatrix} ;
N_2(\varepsilon) = \begin{bmatrix}
\nabla_f L \\
\nabla_q L \\
\mu L \\
q L
\end{bmatrix}
$$

where $\nabla_f L$ and $\nabla_q L$ are the vectors of derivatives of $\nabla_f L$ (see Eq. (16)) and $\nabla_q L$ (see Eq. (17)) with respect to perturbed parameter $\varepsilon$, respectively. $\nabla_\mu L$ and $\nabla_\tau L$ can be expressed as follows:

$$
\nabla_\mu L = \left[\partial \left( \sum_k \sum_{j=k} f_{m(k,j)}^r - q_m^r \right) / \partial \varepsilon, \forall m \right] = 0
$$
\[ \nabla_{\tau L} = \left[ \frac{\partial}{\partial \varepsilon} \left( \sum_{m \in M_r} q_m^{rs} - q^{rs}(\varepsilon) \right) \right]_{\forall r, s, \varepsilon} = -\nabla_{\varepsilon} q^{rs}(\varepsilon) \]  

(42)

4. Numerical example

4.1 Effect of route overlap on modal split and network flow

In this section, network 1 shown in Figure 2 is used to demonstrate the issues with the MNL-MNL model and illustrate how the MNL-PCL model addresses the route overlap problem. It contains one O-D pair, three road routes, and a subway line connecting the origin and destination nodes. The three routes of the road network are: route 1: link 1; route 2: links 2-3, and route 3: links 2-4. The travel time of the subway line from node A to node B is fixed at 5 minutes. The standard Bureau of Public Road (BPR) function is used to characterize link travel times in the road network. Let \( \gamma = 0.5, \varsigma = 1, \theta_{auto} = \theta_{subway} = 1 \). The total travel demand is 10. The parameters for the BPR function and the subway line are shown in Table 1.

The free flow travel time of the overlapped link 2 (i.e., \( x \) as shown in Table 1) is varied from 0.001 to 4.999. An increase in \( x \) will result in higher similarity between routes 2 and 3. Figure 3(a) shows the evolution of the auto and subway demand for the MNL-PCL and MNL-MNL (Oppenheim 1995) models. It demonstrates that the demand split of MNL-MNL model between the auto and subway modes is relatively unchanged as \( x \) increases. Note that when \( x \) increases sufficiently close to 5, the road network will reduce to two routes (links 1 and 2). Thereby, auto mode demand should reduce significantly at \( x = 4.999 \) as one route is removed from the road network. Hence, the estimates of the MNL-MNL model are erroneous. This problem occurs because it cannot account for the route overlap problem, and assigns significant flow to the overlapping routes. To better illustrate the route overlap problem, Figure 3(b) compares the auto flow on routes 1 and 2 estimated by the MNL-MNL model and the MNL-PCL model when \( x \) varies from 0.001 to 4.999. It shows that the auto flows on
routes 1 and 2 estimated by the MNL-MNL model are relatively close despite the increasing similarity between routes 2 and 3. As a result, the auto demand is overestimated as $x$ increases, especially when one route is removed from the road network at $x = 4.999$.

This issue is addressed by the proposed MNL-PCL model. As can be seen in Figure 3(b), the flow on route 2 decreases significantly when the similarity between routes 2 and 3 increases, leading to reduced demand for the auto mode (see Figure 3(a)). The EPT time under the MNL-PCL model increases as the similarity between routes 2 and 3 increases (see Figure 3(c)), because a high value of $x$ will decrease the quality of the road network, forcing drivers to spend more time on the more congested link 2.

4.2 Applications of sensitivity analysis of the MNL-PCL model

This section uses the sensitivity analysis of the MNL-PCL model for several applications including: (1) estimating the perturbed solution of the MNL-PCL model to avoid computational issue when network inputs (e.g., link capacity) are changed due to expected or unexpected events, (2) identifying the critical parameters that significantly impact network performance (e.g., total travel time, total emissions etc.) and travelers’ choice of “go-green” modes, (3) evaluating the effect of link access control on choice of “go-green” modes and flows on other links in the road network, and (4) estimating the distribution of mode-specific O-D demand and link flows under uncertainty in parametric variations. These applications assist planners and decision-makers to design more effective measures to improve the network performance.

Study network 2, shown in Figure 4, is used for addressing the aforementioned applications. It consists of two sub-networks, the auto network and the subway network. The auto network built on the Sioux Falls network consists of 24 nodes, 76 links and 552 O-D pairs. The subway network contains two subway lines from node 1 to node 7, each of which
consists of two subway links. The two subway lines connect 5 O-D pairs including O-D pair 1-6, 6-7, 1-7, 1-4 and 4-7. Let $W_{\text{auto}}$ and $W_{\text{subway}}$ denote set of all O-D pairs in the auto and subway networks, respectively. Note that for this study network, $W_{\text{subway}} \subseteq W_{\text{auto}}$. The following BPR function is used to characterize the travel time of links in the auto and subway networks (Kitthamkesorn et al., 2016):

$$t_{m,a} = t_{m,a}^{0} \left[ 1 + 0.15 \left( \frac{v_{m,a}}{s_{m,a}} \right)^{4} \right], \quad m \in \{ \text{auto, subway} \}$$

The inputs of the BPR function for the auto network (i.e., Sioux Falls network) and the total demand between each O-D pair can be found in Leblanc (1973). The inputs of the subway network are listed in Table 2. The dispersion parameters are: $\theta_{\text{auto}} = \theta_{\text{subway}} = 1$; $\gamma_{rs}^{subway} = \gamma = 0.5$, $\forall rs \in W_{\text{subway}}$ ; $\Psi_{subway}^{rs} = \Psi_{subway} = 2.5$, $\forall rs \in W_{\text{subway}}$ ; $\Psi_{auto}^{rs} = \Psi_{auto} = 2.5$, $\forall rs \in W_{\text{subway}}$ ; $\zeta = 1$.

For path generation in the MNL-PCL model, the revised K-shortest path method developed by De La Barra et al. (1993) is used to find the routes for each O-D pair that are likely to be used by travelers. Only acyclic paths are considered in this study. This method enables us to find around 3400 such routes in the Sioux Falls network.

### 4.2.1 Estimation of perturbed solutions of MNL-PCL model

Planned (e.g., road maintenance) or unexpected (e.g., collapse of bridge, flooding, etc.) events can cause significant changes in network flow and mode-specific O-D demand. Estimating the perturbed network flow and mode-specific O-D demand is important for planners and decision-makers to evaluate the impacts of these events on network performance, and to design effective control strategies to reduce their negative effects (e.g., congestion on some links, lower demand for “go-green” modes). However, solving the MNL-PCL model is computationally expensive due to the PCL model. The first-order linear approximation
(FOLA) method is able to circumvent the computational issue of solving the MNL-PCL model by estimating the perturbed solution with the first-order Taylor approximation. The FOLA method estimates the solution of the flow on link \( a \) (that is, \( \hat{v}_{m,a}, a \in A_m, m \in \{auto, subway\} \)) and demand of mode \( m \) for O-D pair \( r-s \) (that is, \( \hat{q}_{m}^{rs}, m \in \{auto, subway\} \)) after a parametric perturbation as follows:

\[
\begin{align*}
\hat{v}_{m,a} & \approx v_{m,a}^* + \nabla_x v_{m,a} \cdot \delta_x, \quad a \in A_m, m \in \{auto, subway\}, \\
\hat{q}_{m}^{rs} & \approx q_{m}^{rs} + \nabla_x q_{m}^{rs} \cdot \delta_x, \quad rs \in W_m, m \in \{auto, subway\},
\end{align*}
\]

where \( v_{m,a}^* \) and \( q_{m}^{rs} \) are the unperturbed solutions of flow on link \( a \) in network of mode \( m \) (\( a \in A_m, m \in \{auto, subway\} \)) and demand of mode \( m \) for O-D pair \( r-s \), respectively. \( \delta_x \) is the scale of the perturbation of parameter \( \varepsilon \). The derivatives \( \nabla_x q_{m}^{rs}, rs \in W_m, m \in \{auto, subway\} \) can be obtained directly using Eq. (40). Let \( \Delta = \{d_{a,m,k}, \forall a, m, k\} \) be the link-path incidence matrix, where \( d_{a,m,k} = 1 \) if the \( k \)th path in vector \( \bar{f} \) uses link \( a \) in network of mode \( m \). The derivatives \( \nabla_x v_{m,a}, a \in A_m, m \in \{auto, subway\} \), can be calculated by

\[
[\nabla_x v_{m,a}, \forall a \in A_m, m \in \{auto, subway\}] = \Delta \cdot \nabla_x \bar{f}
\]

where \( \nabla_x \bar{f} \) can be obtained directly using Eq. (40).

In the numerical example, we assume the total demand between the five O-D pairs in set \( W_{subway} \) is increased by 10% simultaneously. Figure 5 shows the relative error of estimated link flow on the ten links with the poorest estimation performance and the relative error of estimated auto and subway demand for the five O-D pairs. It shows that the FOLA provides very good estimations of link flow and mode-specific demands, with the relative errors being less than 7%. Hence, the FOLA method is effective in approximating the variations in link flows and mode-specific O-D demand caused by disruptive events. It should be noted that the
accuracy of the estimated equilibrium solution depends on the scale of the perturbation; the larger the perturbation, the lower the estimation accuracy of the FOLA method.

4.2.2 Critical parameter analysis

To design effective strategies to improve network performance (e.g., reduce congestion), decision-makers are often faced with two problems: (i) how can network inputs be controlled so that network performance can be improved? and (ii) how to select parameters whose control can enhance network performance? These two problems can be addressed using critical parameter analysis. A critical parameter in a traffic assignment model is defined as a measurable attribute that has a strong impact on the equilibrium solution. This study seeks to identify the critical parameters by calculating the elasticity of the network performance indicators with respect to the perturbed parameters. Elasticity is a dimensionless measure of the response of one variable to another (Du, Cheng, and Rakha 2012). It measures the percent change in output due to a one percent change in input. The elasticity (denoted as $E^\varepsilon_y$) of network performance measure $y$ with respect to perturbed parameter $\varepsilon$ is:

$$E^\varepsilon_y = \frac{\partial y}{\partial \varepsilon} \cdot \frac{\varepsilon}{y}.$$ 

Decision-makers often seek to design strategies to promote “go-green” modes to reduce congestion and improve the environment. Hence, three related network performance indicators are considered in this study, including total subway demand (TSD), total emissions (TE) and total travel time (TTT), as follows:

$$TSD = \sum_{r,s,w} q_{\text{subway}}^{rs}$$

$$TE = \sum_{a,v} v_{\text{auto},a} \eta_a$$

$$TTT = \sum_{a,v} t_{\text{auto},a} v_{\text{auto},a} + \sum_{a,v} t_{\text{subway},a} v_{\text{subway},a}$$
where \( \eta_a \) is the emission factor associated with link \( a \). For simplicity, we set \( \eta_a = 1 \) (Du, Cheng, and Rakha 2012). The elasticities of the three performance indicators with respect to the perturbed parameters are:

\[
E_{\text{TSD}}^e = \frac{\partial \text{TSD}}{\partial e} \frac{e}{\text{TSD}} = \left( \sum_{rs \in \text{subway}} \frac{\partial q_{\text{subway}}^r}{\partial e} \right) \frac{e}{\sum_{rs \in \text{subway}} q_{\text{subway}}^r}
\]

\[
E_{\text{TE}}^e = \frac{\partial \text{TE}}{\partial e} \frac{e}{\text{TE}} = \frac{e}{\text{TE}} \sum_{a \in \text{auto}} \frac{\partial v_{\text{auto},a}}{\partial e} \eta_a
\]

\[
E_{\text{TTT}}^e = \frac{\partial \text{TTT}}{\partial e} \frac{e}{\text{TTT}} = \frac{e}{\text{TTT}} \left\{ \sum_{m \in \{\text{auto, subway}\}} \sum_{a \in A_m} \left[ (t_{m,a} + 0.15 \cdot t_{m,a}^0 \cdot 4 \cdot \left( \frac{v_{m,a}}{s_{m,a}} \right)^3) \frac{\partial v_{m,a}}{\partial e} \right] \right\}
\]

where the terms \( \frac{\partial v_{m,a}}{\partial e}, m \in \{\text{auto, subway}\} \) and \( \frac{\partial q_{\text{subway}}^r}{\partial e}, rs \in \text{W}_{\text{subway}} \) in above equations can be obtained using Eq. (43) and Eq. (40), respectively.

Table 3 presents the elasticities of the three performance indicators with respect to the parameters for exogenous attractiveness (\( \Psi_{\text{auto}}, \Psi_{\text{subway}} \)), the O-D specific discrete parameter (\( \gamma \)) and the parameters for free flow travel time of subway links (\( t_{\text{subway},1}^0, t_{\text{subway},2}^0, t_{\text{subway},3}^0, t_{\text{subway},4}^0 \)). It shows that among these parameters, the free flow travel time of subway links is the most important factor for the three performance indicators. A 1% reduction in the free flow travel time of subway link 1 can increase TSD by 0.401%, reduce TE by 0.0021% and decrease TTT by \( 2.33 \times 10^{-7} \% \). The results also illustrate that an increase in the exogenous attractiveness of the auto mode (i.e., \( \Psi_{\text{auto}} \)) will increase TSD. This is counterintuitive, and occurs because an increase in \( \Psi_{\text{auto}} \) will attract more demand to the auto road network, which increase congestion on the auto road network. Thereby, more demand is shifted from auto to subway. These results suggest that decision-makers should systematically and holistically
analyze the impacts of designed control strategies on network performance. Critical parameter analysis can help to prevent the failure of or ineffective design of network improvement control strategies.

4.2.3 Access control
Access control (such as lane closure, toll charge on links, etc.) is often deployed in areas such as congested inner cities, residential areas and school zones, to improve safety and environmental conditions by restricting through trips. Here, the numerical example explores how to design effective access control strategies to reduce the through link flow in an area and its impact on the “go-green” mode demand. Table 4 shows the derivatives of equilibrium link flows and mode-specific O-D demand with respect to free flow travel time of five links. Note the free flow travel time of links can be controlled using speed limit. Table 4 illustrates that reducing the speed limit of certain links may not be the most effective way to reduce through traffic on them. For example, suppose link 4 is located in front of a school zone and it is desirable to reduce flow through them during a certain time period. Then, controlling link 1’s free flow travel time can have more impact on the flow of link 4 than controlling free flow travel time of link 4. Table 4 also shows that access control on link 1 will significantly increase subway demand for O-D pairs 1-6, 6-7, and 1-7. These results help to evaluate how demand shifts from the auto mode to other modes.

4.2.4 Uncertainty analysis
A transportation system is typically subject to uncertainties; for example, traffic demand usually fluctuates over time. Considering the impacts of such uncertainties on the equilibrium solution is critical for the accurate evaluation of project and investment decisions. Systematic uncertainty analysis seeks to address this problem by providing insights on the level of confidence in model outputs, and by identifying the critical sources of uncertainty to enhance
robustness of the travel demand model (Yang et al. 2013). It is performed by leveraging the sensitivity analysis results of the MNL-PCL model. Consider that an input parameter $\varepsilon$ is associated with uncertainty and is subject to variation with a standard deviation $\sigma_\varepsilon$. Then, the standard error of the outputs $y$ (i.e., link flows and O-D specific demand) can be estimated as follows:

$$\sigma_y^2 = \nabla_\varepsilon y \cdot \sigma_\varepsilon^2 \cdot \nabla_\varepsilon y^T$$ (43)

where $\nabla_\varepsilon y$ can be obtained using Eq. (40). Eq. (43) can be used to determine confidence intervals for link flows and mode-specific O-D demand if the probability distribution function of $\varepsilon$ is given. In the numerical example, we assume that the total demand for O-D pair 1-7 ($q^{17}$) is subject to uncertainty over time with a normal distribution $N(500,10000)$. Then, the variance of model outputs will be estimated using Eq.(43). For analysis purposes, the value of $q^{17}$ is randomly generated 1200 times according to the normal distribution $N(500,10000)$, and the MNL-PCL model is solved 1200 times using the sampled $q^{17}$. The mean and standard deviation of the model outputs are obtained from the simulated results. Table 5 compares the estimated and simulated results. It illustrates that the estimated variances of link flows and demands of different modes are very close to those of the simulated results, which helps to accurately estimate the 90% confidence interval of link flows and mode-specific O-D demand.

5. Concluding comments
This study develops a CMSTA model in which an MNL model is used to characterize travelers’ mode choice decisions and a PCL model is used to characterize travelers’ route choice decisions. The PCL model can address the route overlap problem by evaluating the similarity of each pair of routes independently using a similarity index. This enables the proposed MNL-PCL model to better capture travelers’ mode and route choices. In addition, in contrast to the CNL model, the PCL model does not involve nest-specific coefficients which
add significant computational complexity to parametric calibration. Hence, the MNL-PCL model is more tractable in practice. The study also develops an analytical model for sensitivity analysis of the MNL-PCL model. The application of the MNL-PCL model in a numerical example illustrates that it overcomes the route overlap problem and is able to better estimate mode-specific O-D demand and road network flows. The applications addressed by leveraging the sensitivity analysis of the MNL-PCL model indicate that the sensitivity analysis can aid the more accurate estimation of perturbed link flows and mode-specific O-D demand. Further, it can help to identify critical parameters (e.g., subway travel time in this study) to improve network performance, evaluate the impacts of access control measures on link flows and modal split, and provide confidence intervals for mode-specific O-D demand when O-D demand varies over time. These applications can assist planners and decision-makers to design more efficient and effective strategies to promote “go-green” travel modes, and reduce network congestion and green-house gas emissions.

This study can be extended in a few directions. First, due to the independence assumption embedded in the MNL model for modal split, the MNL model cannot account for the unobserved similarities in different traffic modes. To address this problem, the nested logit model or PCL model can be used to characterize travelers’ mode choice behavior. Second, the PCL model accounts for not only the route overlap problem, but can also be scaled to account for perception variance related to different trip lengths (Gliebe, Koppelman, and Ziliaskopoulos 1999). Thereby, a scaling technique can be developed and incorporated in the proposed model.

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References


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<table>
<thead>
<tr>
<th>Links</th>
<th>Mode</th>
<th>Free flow travel time ($t_{m,a}^0$)</th>
<th>Link capacity ($s_{m,a}$)</th>
<th>Exogenous attractiveness ($\Psi^r_m$)</th>
<th>Travel time</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Auto</td>
<td>5</td>
<td>6</td>
<td>1.5</td>
<td>$t_{auto,a} = t_{auto,a}^0 \left[ 1 + 0.15 \left( \frac{v_{auto,a}}{s_{auto,a}} \right)^4 \right]$</td>
</tr>
<tr>
<td></td>
<td>Subway</td>
<td>5</td>
<td>6</td>
<td>2.5</td>
<td>Constant</td>
</tr>
</tbody>
</table>

Table 1. Inputs for study network 1
Table 2. Inputs for subway network in study network 2

<table>
<thead>
<tr>
<th>Subway lines</th>
<th>Subway line 1</th>
<th>Subway line 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Links</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Free flow travel time ($t_{subway,a}^0$)</td>
<td>11</td>
<td>16</td>
</tr>
<tr>
<td>Capacity ($s_{subway,a}$)</td>
<td>3000</td>
<td>3000</td>
</tr>
</tbody>
</table>
Table 3. Elasticities of $TD$, $TE$, and $TTT$ with respect to perturbed parameters

<table>
<thead>
<tr>
<th></th>
<th>$\Psi_{\text{car}}$</th>
<th>$\Psi_{\text{transit}}$</th>
<th>$\gamma$</th>
<th>$\gamma_{\text{subway},1}$</th>
<th>$\gamma_{\text{subway},2}$</th>
<th>$\gamma_{\text{subway},3}$</th>
<th>$\gamma_{\text{subway},4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TSD$</td>
<td>0.1076</td>
<td>0.2726</td>
<td>-0.3296</td>
<td>-0.4046</td>
<td>-0.6893</td>
<td>-0.3416</td>
<td>-0.1158</td>
</tr>
<tr>
<td>$TE$</td>
<td>-0.00062</td>
<td>-0.0011</td>
<td>0.00134</td>
<td>0.00205</td>
<td>0.00315</td>
<td>0.00124</td>
<td>0.00081</td>
</tr>
<tr>
<td>$TTT$</td>
<td>-0.9518</td>
<td>-1.1203</td>
<td>1.3656</td>
<td>2.3277</td>
<td>5.5021</td>
<td>0.1816</td>
<td>1.2557</td>
</tr>
</tbody>
</table>

($\times 10^7$)
Table 4. Derivatives of link flows and mode-specific demands with respect to capacities of road links

<table>
<thead>
<tr>
<th></th>
<th>$t^0_{\text{auto},1}$</th>
<th>$t^0_{\text{auto},4}$</th>
<th>$t^0_{\text{auto},6}$</th>
<th>$t^0_{\text{auto},14}$</th>
<th>$t^0_{\text{auto},20}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_{\text{auto},1}$</td>
<td>-134.56</td>
<td>-130.56</td>
<td>81.01</td>
<td>2.11</td>
<td>6.29</td>
</tr>
<tr>
<td>$v_{\text{auto},4}$</td>
<td>-195.65</td>
<td>-194.30</td>
<td>123.74</td>
<td>-2.04</td>
<td>8.40</td>
</tr>
<tr>
<td>$v_{\text{auto},6}$</td>
<td>86.57</td>
<td>92.94</td>
<td>-167.19</td>
<td>1.57</td>
<td>-11.55</td>
</tr>
<tr>
<td>$v_{\text{auto},14}$</td>
<td>3.01</td>
<td>-2.36</td>
<td>2.35</td>
<td>-127.01</td>
<td>-2.59</td>
</tr>
<tr>
<td>$v_{\text{auto},20}$</td>
<td>10.6</td>
<td>10.01</td>
<td>-17.93</td>
<td>-2.59</td>
<td>-255.12</td>
</tr>
<tr>
<td>$q_{\text{subway}}^{16}$</td>
<td>25.59</td>
<td>-6.04</td>
<td>23.88</td>
<td>0.20</td>
<td>0.18</td>
</tr>
<tr>
<td>$q_{\text{subway}}^{67}$</td>
<td>36.37</td>
<td>-8.44</td>
<td>34.03</td>
<td>0.30</td>
<td>0.23</td>
</tr>
<tr>
<td>$q_{\text{subway}}^{17}$</td>
<td>3.98</td>
<td>3.17</td>
<td>6.70</td>
<td>63.31</td>
<td>-0.49</td>
</tr>
<tr>
<td>$q_{\text{subway}}^{14}$</td>
<td>-0.10</td>
<td>0.34</td>
<td>0.13</td>
<td>-0.01</td>
<td>-0.04</td>
</tr>
<tr>
<td>$q_{\text{subway}}^{47}$</td>
<td>0.42</td>
<td>9.76</td>
<td>7.57</td>
<td>-0.04</td>
<td>2.12</td>
</tr>
</tbody>
</table>
Table 5. Confidence interval of the equilibrium demands and flow on the five links which impacted the most in terms of flow when $q^{17}$ has uncertainty

<table>
<thead>
<tr>
<th>Solution variable</th>
<th>Variance (simulated)</th>
<th>Variance (estimated)</th>
<th>90% confidence interval</th>
<th>5%</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Simulated</td>
<td>Estimated</td>
<td>Simulated</td>
</tr>
<tr>
<td>$v_{auto,3}$</td>
<td>11.4954</td>
<td>11.6426</td>
<td>4669.9130</td>
<td>4669.6000</td>
<td>4707.729</td>
</tr>
<tr>
<td>$v_{auto,4}$</td>
<td>11.4126</td>
<td>11.5622</td>
<td>6479.4940</td>
<td>6479.1900</td>
<td>6517.038</td>
</tr>
<tr>
<td>$v_{auto,6}$</td>
<td>3.0472</td>
<td>3.0655</td>
<td>11143.0100</td>
<td>11143.2900</td>
<td>11153.04</td>
</tr>
<tr>
<td>$v_{auto,14}$</td>
<td>0.0452</td>
<td>0.0425</td>
<td>7102.2140</td>
<td>7102.1290</td>
<td>7102.363</td>
</tr>
<tr>
<td>$v_{auto,20}$</td>
<td>2.2612</td>
<td>2.2739</td>
<td>7248.3560</td>
<td>7248.1700</td>
<td>7255.795</td>
</tr>
<tr>
<td>$q_{auto}^{16}$</td>
<td>0.4548</td>
<td>0.4595</td>
<td>88.6682</td>
<td>88.7098</td>
<td>90.1642</td>
</tr>
<tr>
<td>$q_{auto}^{67}$</td>
<td>1.4206</td>
<td>1.4469</td>
<td>154.4532</td>
<td>154.5502</td>
<td>159.1265</td>
</tr>
<tr>
<td>$q_{auto}^{17}$</td>
<td>14.9887</td>
<td>15.1427</td>
<td>59.0605</td>
<td>58.5928</td>
<td>108.3689</td>
</tr>
<tr>
<td>$q_{auto}^{14}$</td>
<td>0.0078</td>
<td>0.0062</td>
<td>273.8471</td>
<td>273.8410</td>
<td>273.8726</td>
</tr>
<tr>
<td>$q_{auto}^{47}$</td>
<td>0.1287</td>
<td>0.1355</td>
<td>11.4598</td>
<td>11.4342</td>
<td>11.8831</td>
</tr>
<tr>
<td>$q_{subway}^{16}$</td>
<td>0.4548</td>
<td>0.4595</td>
<td>209.8358</td>
<td>209.7786</td>
<td>211.3318</td>
</tr>
<tr>
<td>$q_{subway}^{67}$</td>
<td>1.4206</td>
<td>1.4469</td>
<td>240.8735</td>
<td>240.6900</td>
<td>245.5468</td>
</tr>
<tr>
<td>$q_{subway}^{17}$</td>
<td>84.0803</td>
<td>84.9573</td>
<td>281.8069</td>
<td>276.7573</td>
<td>558.4065</td>
</tr>
<tr>
<td>$q_{subway}^{14}$</td>
<td>0.0078</td>
<td>0.0062</td>
<td>226.1274</td>
<td>226.1385</td>
<td>226.1529</td>
</tr>
<tr>
<td>$q_{subway}^{47}$</td>
<td>0.1287</td>
<td>0.1355</td>
<td>388.1169</td>
<td>388.12</td>
<td>388.5402</td>
</tr>
</tbody>
</table>
Figure 1. Illustration of the hierarchical structure of the PCL model.
Figure 2. Study network 1.
Figure 3. (a) Evolution of auto and subway demand estimated according to the MNL-MNL and MNL-PCL models; (b) Comparison of auto flow on routes 1 and 2 estimated by the two models; (c) Evolution of EPT time of road network for the two models
FIGURE 4. Study network 2.
Figure 5. (a) Relative error in estimated flows for the ten links with poorest estimation performance on the auto network; (b) Relative errors for the auto and subway demand for the five O-D pairs that are connected using both the auto and subway modes.