An extended microscopic traffic flow model based on the spring-mass system theory

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This study proposes a new microscopic traffic flow model based on the spring-mass system theory. In particular, considering the similarity between the acceleration or deceleration behavior in traffic flow and the scaling properties of a spring, a car-following (CF) model is proposed based on the fundamental physical law of the spring-mass system. Stability of the proposed model is analyzed using the perturbation method to obtain the stability condition. Numerical experiments are performed through simulation. The results demonstrate the proposed model can capture the characteristic of propagation backwards of disturbance in traffic flow. In addition, the findings of this study provide insights in modeling traffic flow from the mechanical system theory perspective.

Keywords: Car-following model; traffic flow; spring-mass system theory; stability analysis.

1. Introduction

Traffic problems, i.e. traffic congestion, traffic management, and traffic emissions, have been widely concerned and investigated in recent years. To formulate the complex mechanisms behind the phenomena of vehicular traffic flow, various traffic flow models have been developed to capture the behavior of vehicle movement. Generally speaking, the existing traffic flow models can be categorized as microscopic and macroscopic models. From the perspective of microscopic traffic models, an individual vehicle is represented by a particle and consequently the vehicular traffic flow is treated as a system of interacting particles,†–10 and therefore microscopic
traffic flow models generally use the microscopic variables including velocity, position, and acceleration to capture the local interactions between vehicles. From the perspective of macroscopic traffic models, the vehicular traffic flow is viewed as a compressible fluid formed by vehicles, and macroscopic traffic flow models mainly use the collective variables such as the density, volume and average speed to characterize the traffic flow accordingly.

Microscopic models can be further classified as car-following (CF) models or cellular automation (CA) models. CF models describe the interactions with preceding vehicles in the same lane based on the idea that each driver controls a vehicle under the stimuli from the preceding vehicle, such as the Gazis–Herman–Rothery (GHR) model and its variations, Gipps model, optimal velocity (OV) model and its variations, intelligent driver model, fuzzy-logic model, as well as psycho-physical models.

On the other hand, CA models describe the traffic phenomenon using a stochastic discrete approach, such as Rule 184 model, Biham–Middleton–Levine (BML) model, as well as Nagel–Schreckenberg (NaSch) model and its variations.

Similarly, macroscopic models can be further classified as kinematic models, dynamical models, anisotropic models, as well as lattice hydrodynamic models. Kinematic models describes traffic flow as continuum fluid flow. The kinematic wave model is Lighthill–Whitham–Richards (LWR) model. LWR model relies on the assumption that the movement of vehicles satisfies the equilibrium speed–density relationship, which does not cover the characteristics of traffic flow under the non-equilibrium state. Hence, a few variations of LWR model are developed to address this issue. Dynamical models are represented by the Payne–Whitham (PW) model, which captures the stop-and-go wave in time and space. Later, several variations of PW model are proposed with emphasis on the momentum equation. Anisotropic models can date back to the model proposed by Daganzo, which focuses on the anisotropic characteristics of traffic. Lattice hydrodynamic models are first proposed by Nagatani, in which the concept of discrete lattice is introduced to derive the modified Korteweg de Vries (mKdV) equation to characterize the density wave profile.

Note that the above mentioned traffic flow models are based on the fluid-dynamic theory. However, the acceleration or deceleration of vehicles in traffic flow behaves like a mechanical element, i.e. spring-mass element. For example, the following vehicle will accelerate when the space headway between the leading vehicle and the following vehicle exceeds the critical safe space headway. While the following vehicle will decelerate when the space headway unexceeds the critical safe space headway. This phenomenon is similar to the characteristics of mechanical systems. Hence, there is a research need to study the characteristics of traffic flow from the mechanical system perspective.

The purpose of this study is to study the microscopic vehicular traffic flow behavior from the mechanical system theory perspective. In particular, note that the space headway and the velocity difference in the vehicular traffic flow are coupled.
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That is to say, when the space headway between the leading vehicle and the following vehicle is large, the following vehicle will accelerate, and consequently it will result in a small headway. When the space headway approaches the critical safe value, the following vehicle needs to reduce its acceleration to keep safe space headway. It implies that the actual space headway between the leading and following vehicles fluctuates around the safe space headway. This characteristic of vehicular traffic flow is similar to the scaling properties of the spring-mass system. Specially, the critical safe space headway between the leading and following vehicles can be viewed as the equilibrium length of the spring in the spring-mass system. Hence, motivated by the similarity between the traffic system and the spring-mass system, this study proposes a CF model based on the fundamental physical law of the spring-mass system. Stability of the proposed model is analyzed using the perturbation method to obtain the stability condition. Numerical experiments are performed through simulation. The results demonstrate the effectiveness of the proposed model in terms of the smoothness and perturbation rejection.

The rest of this paper is organized as follows. Section 2 proposes a new CF model based on the spring-mass system theory. Section 3 performs the stability analysis of the proposed model using the perturbation method. Section 4 conducts the simulation-based numerical experiments. The final section concludes this study.

2. Model Derivation

Figure 1 shows the scenario of CF process between the leading vehicle and the following vehicle in the same lane. In this scenario, the leading vehicle (i.e. vehicle \( n + 1 \)) is traveling in front of the following vehicle (i.e. vehicle \( n \)). The behavior of the following vehicle is associated with the stimulus from the leading vehicle, such as the position, velocity, and space headway as well as velocity difference. Hence, the CF model can be formulated as follows:

\[
a_n(t) = f(x_n(t), v_n(t), \Delta x_n(t), \Delta v_n(t)) ,
\]

Fig. 1. Car-following model.
where $x_n(t)$, $v_n(t)$ and $a_n(t)$ represent the position, velocity, and acceleration of the vehicle $n$ at time $t$, respectively; $\Delta x_n(t) = x_{n+1}(t) - x_n(t)$ and $\Delta v_n(t) = v_{n+1}(t) - v_n(t)$ represent the space headway and velocity difference between the leading vehicle (i.e. vehicle $n+1$) and the following vehicle (i.e. vehicle $n$). $f$ is a function to be determined according to the specific traffic condition.

Following this idea, various CF models have been developed by factoring the surroundings of the following vehicle, including generalized force (GF) model, full velocity difference (FVD) model, multiple ahead and velocity difference model, full velocity and acceleration difference model, multiple velocity difference model, and multiple headway, velocity and acceleration difference model. However, the above-mentioned CF models are derived from the fluid-dynamic perspective. Unlike the existing CF models, this study focuses on the interaction between the leading vehicle and the following vehicle from the mechanical system theory perspective. As shown in Fig. 2, we assume a spring exists in the CF process. Considering the similarity between the acceleration or deceleration behavior in the vehicular traffic flow system and the scaling properties of the spring in the spring-mass dynamic system, spring-mass system theory is used to analogize the acceleration or deceleration behavior in the vehicular traffic. Therefore, according to the Hooke’s law in spring-mass system, the dynamic model of traffic flow can be formulated as follows:

$$ma_n(t) = k(\Delta x_n(t) - L),$$

where $m, k, L \in \mathbb{R}$ and $m, k, L > 0$ represent the mass of the vehicle $n$, spring stiffness coefficient and the spring equilibrium length, respectively. $a_n(t)$ denotes the acceleration of the vehicle $n$ at time $t$. $\Delta x_n(t) = x_{n+1}(t) - x_n(t)$ represents the space headway between the vehicle $n$ and the vehicle $n+1$ at time $t$, where $x_n(t)$ and $x_{n+1}(t)$ denote the positions of the following vehicle $n$ and the leading vehicle $n+1$ at time $t$, respectively.

For convenience, Eq. (2) can be rewritten as

$$a_n(t) = \lambda(\Delta x_n(t) - L),$$

where $\lambda = \frac{k}{m}$. It can be called as the sensitivity coefficient in the proposed model.
3. Asymptotic Stability Analysis

Regarding CF models, a stable state indicates that all vehicles can move free with the safe velocity and distance. And the asymptotic stability analysis of traffic flow concerns the attenuation or increase of the amplitude of perturbation as it propagates toward downstream traffic. The stability analysis of the proposed model is performed based on the following assumption.

Assumption 1. The initial state of the traffic flow is steady, and all vehicles in the traffic flow move with the identical space headway $b$ and the constant velocity $v_0$.

Based on the assumption, the following theorem can be obtained accordingly.

**Theorem 1.** If the difference between the initial space headway and the critical safe space headway satisfies:

$$\delta = b - L \leq 6,$$

then the proposed model (3) is stable.

**Proof.** Following Assumption 1, the position solution to the steady flow is

$$x_n^0(t) = bn + v_0 t,$$

where $x_n^0(t)$ is the position of the $n$th vehicle in steady state.

Adding a small disturbance $y_n(t)$ to the steady-state solution $x_n^0(t)$, i.e.

$$y_n(t) = x_n(t) - x_n^0(t).$$

Substituting Eq. (6) into Eq. (3) and linearizing the resulting equation using the Taylor expansion, it deduces:

$$\ddot{y}_n(t) = \lambda (\Delta y_n(t) + b - L).$$

Let $\delta = b - L$ and set $y_n(t)$ in the Fourier models, i.e. $y_n(t) = A \exp(i\alpha n + zt)$, substituting it in Eq. (7) and it follows:

$$z^2 e^{i\alpha n + zt} = \lambda [e^{i\alpha n + zt}(e^{i\alpha} - 1) + \delta].$$

Note that $\exp(i\alpha) = \cos \alpha + i \sin \alpha$ using the Euler’s formula and let $z = r + iw$ with $r = 0$. It follows from Eq. (8) that

$$-w^2[\cos(\alpha n + wt) + i \sin(\alpha n + wt)] + \lambda[\cos(\alpha n + wt)] + i \sin(\alpha n + wt)](\cos \alpha - 1 + i \sin \alpha) + \lambda \delta = 0.$$

Denote $\sigma_c = \cos(\alpha n + wt), \sigma_s = \sin(\alpha n + wt)$. Then Eq. (9) can be rewritten as

$$-w^2(\sigma_c + i\sigma_s) + \lambda(\sigma_c + i\sigma_s)(\cos \alpha - 1 + i \sin \alpha) + \lambda \delta = 0.$$
Equivalently
\[-w^2\sigma_c - iw^2\sigma_s + \lambda[\sigma_c(\cos \alpha - 1) - \sigma_s \sin \alpha] + i\lambda[\sigma_s(\cos \alpha - 1) + \sigma_c \sin \alpha] + \lambda\delta = 0.\]  
(11)

Consequently, we have
\[-w^2\sigma_c + \lambda[\sigma_c(\cos \alpha - 1) - \sigma_s \sin \alpha] + \lambda\delta = 0\]
\[-w^2\sigma_s + \lambda[\sigma_s(\cos \alpha - 1) + \sigma_c \sin \alpha] = 0.\]  
(12)

Based on Eq. (12), we can obtain
\[w^2 = \frac{\lambda[\sigma_s(\cos \alpha - 1) + \sigma_c \sin \alpha]}{\sigma_s}.\]  
(13)

Then
\[\delta = \frac{\sigma_s(\cos \alpha - 1) + \sigma_c \sin \alpha - \sigma_s[\sigma_c(\cos \alpha - 1) - \sigma_s \sin \alpha]}{\sigma_s} = \frac{\sigma_s(\cos \alpha - 1)(1 - \sigma_c) + \sigma_c \sin \alpha + \sigma_s^2 \sin \alpha}{\sigma_s}.\]  
(14)

Note that |\sin \alpha| \leq 1 and |\cos \alpha| \leq 1, we can have
\[|\delta| \leq \left| \frac{\sigma_s(\cos \alpha - 1)(1 - \sigma_c) + \sigma_c \sin \alpha + \sigma_s^2 \sin \alpha}{\sigma_s} \right| \leq \frac{|\sigma_s(\cos \alpha - 1)(1 - \sigma_c) + |\sigma_c \sin \alpha| + |\sigma_s^2 \sin \alpha|}{|\sigma_s|} \leq 6.\]  
(15)

Remark. Unlike the existing CF models, the stability of the proposed model does not depend on the sensitivity coefficient but the difference between the initial space headway and the critical safe space headway. Moreover, according to Eq. (15), the proposed model is stable if \(\delta \leq 6\).

4. Numerical Experiments

Based on the foregoing theoretical analyses, numerical experiments are performed to verify the proposed model described by Eq. (3) and demonstrate its dynamic performance. The experiments are carried out under the same condition as in Ref. 2. Suppose that there are \(N = 100\) vehicles distributed homogeneously on a road under a periodic boundary condition. Note that the first vehicle in the traffic flow is assumed as the leading vehicle in this section. The road length is \(R = 200\) m. The initial conditions are set as follows:\(^2\)
\[x_1^{(0)} = 2m, \quad x_n^{(0)} = (n - 1)b, \quad b = R/N \quad (n = 2, 3, \ldots, N).\]  
(16)

The relevant parameters are chosen as follows:
\[\lambda = 0.015 \text{ s}^{-2}, \quad v_0 = 0.964 \text{ m/s}, \quad L = 2 \text{ m}.\]  
(17)
A small disturbance is added to the first vehicle and the other vehicles keep the same condition. Therefore, we can obtain that

$$x_1(0) = x_1^{(0)} + 0.1; \quad x_n(0) = x_n^{(0)}; \quad v_n(0) = v_n^{(0)}.$$  \hspace{1cm} (18)

The position and velocity of the vehicle $n$ at the time interval $m$ are calculated as follows:

$$x_n(m) = x_n(m - 1) + v_n(m - 1) + \frac{1}{2}a_n(m - 1)t^2 \hspace{1cm} (19)$$

$$v_n(m) = v_n(m - 1) + a_n(m - 1)t, \hspace{1cm} (20)$$

where $n = 1, 2, \ldots, N; \ m = 1, 2, \ldots, \infty$.

The boundary conditions are defined as follows:

$$x_{101}(m) = R + x_1(m), \quad v_{101}(m) = v_1(m). \hspace{1cm} (21)$$

Figure 3 shows the velocity profile of traffic flow at $t = 40$ s, $t = 60$ s, $t = 80$ s and $t = 100$ s, respectively. Particularly, Fig. 2(a) indicates that the velocity of vehicles tends to be stable around the constant velocity $v_0 = 0.964$ m/s. As time goes on, a fluctuation of velocity emerges and the amplitude of the fluctuation is gradually enlarged in the downstream traffic as shown in Figs. 3(b)–3(d). The state evolution accounts for the impact of the disturbance from the leading vehicle on following

![Figure 3. Velocity profile of traffic flow: (a) $t = 40$ s; (b) $t = 60$ s; (c) $t = 80$ s; (d) $t = 100$ s.](image-url)
vehicles. Specifically, at the initial stage, the impact of the disturbance on the traffic flow is not significant, which may be attributed to the sustainable and maintainable capability of the vehicle platoon. And then, when the disturbance is propagated to downstream vehicles, the impact of the disturbance is gradually enlarged. That is because the moving vehicles are affected due to surrounding stimuli. However, the disturbance remains locally and the rest of traffic flow can keep stable. The results of the simulation verify that the proposed model can capture the characteristic of propagation backwards of disturbance in traffic flow.

5. Conclusion

Differing from the viewpoints of existing traffic flow models, this study proposes a new traffic flow model based on the fundamental physical law of the spring-mass system from the mechanical system theory perspective. Stability analysis of the proposed model is performed using the perturbation method. The stability condition of the proposed model is associated with the difference between the initial space headway and the critical safe space headway, but it is independent of the sensitivity coefficients. Results from numerical experiments demonstrate that the proposed model can capture the characteristic of propagation backwards of disturbance in traffic flow.

The study is limited to a relative simple numerical experiment to investigate the characteristic of traffic flow. However, the findings of this study provide insights in modeling traffic flow from the mechanical system theory perspective. Also, it motivates to study the traffic flow behavior such as stop-and-go and platoon formation based on the proposed traffic flow model.

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