A Marginal Utility Day-to-Day Traffic Evolution Model based on One-Step Strategic Thinking

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Abstract

Most deterministic day-to-day traffic evolution models, either in continuous-time or discrete-time space, have been formulated based on a fundamental assumption on driver route choice rationality where a driver seeks to maximize her/his marginal benefit defined as the difference between the perceived route costs. The notion of rationality entails the exploration of the marginal decision rule from economic theory, which states that a rational individual evaluates his/her marginal utility, defined as the difference between the marginal benefit and the marginal cost, of each incremental decision. Seeking to analyze the marginal decision rule in the modeling of deterministic day-to-day traffic evolution, this paper proposes a modeling framework which introduces a term to capture the marginal cost to the driver induced by route switching. The proposed framework enables to capture both benefit and cost associated with route changes. The marginal cost is then formulated upon the assumption that drivers are able to predict other drivers’ responses to the current traffic conditions, which is adopted based on the notion of strategic thinking of rational players developed in behavior game theory. The marginal cost based on 1-step strategic thinking also describes the “shadow price” of shifting routes, which helps to explain the behavioral tendency of the driver perceiving the cost-sensitivity to link/route flows. After developing a formulation of the marginal utility day-to-day model, its theoretical properties are analyzed, including the invariance property, asymptotic stability, and relationships with the rational behavioral adjustment process.

Keywords: Day-to-day traffic evolution; Marginal decision rule; Strategic thinking; Behavioral game theory; Rationality.
1. Introduction

1.1 Motivation

Over the past three decades, a large number of day-to-day (DTD) traffic evolution models have been developed to represent how traffic flow evolves under disequilibrium. These DTD models describe drivers’ individual route switching behavior, and the corresponding traffic pattern changes at an aggregate level. The traffic evolution processes characterized by these DTD models allow transportation planners and operational managers to evaluate the transportation network performance under disequilibrium and help them develop reliable network design and traffic control plans to respond to expected and unexpected transportation network disruptions. Existing DTD models can be classified in terms of whether they are constructed in continuous or discrete time space, and whether they capture the uncertainty in transportation systems (Watling, 1999). Reviews of DTD traffic evolution modeling can be found in Watling and Cantarella (2013).

Most deterministic DTD traffic evolution models, either in continuous-time or discrete-time space, have been formulated based on a fundamental assumption of driver route choice rationality; namely, drivers tend to switch to the routes with lower perceived travel costs. In this context, the difference in perceived travel costs can be viewed as the marginal benefit to the driver. The rate of route switching is then linked to the marginal benefit measured by the travel cost reduction. Hence, in the existing DTD models, the marginal benefit is the basis for a rational individual to switch to an alternative route because she/he perceives that the target route has a cost lower than her/his current one. In this sense, drivers stop route switching when they perceive that the marginal benefit is zero.

Modeling the marginal benefit in classical deterministic DTD traffic evolution models relies on an underlying assumption of “perfect knowledge” on the link/route costs. This implicitly assumes knowledge of the route decisions of other drivers and the link performance functions. The latter reflects that drivers perceive the link/route cost sensitivity to flow through their long-term travel experience, as discussed in Kumar and Peeta (2015). The explicit consideration of link/route cost sensitivity to flow illustrates a marginal cost to the driver that arises from the concurrent decisions of other drivers, which can reduce the anticipated marginal benefit for that driver. In economics, this type of cost represents the “shadow price” of route switching. This marginal cost in DTD traffic evolution context corresponds to the marginal cost in transportation
economics, where an additional vehicle in traffic imposes a definite cost on all users (Mohring, 1976). In this context, a rational driver decides to switch to an alternative route on the next day because she/he perceives that the route cost reduction (i.e., marginal benefit) will be greater than the route cost increase due to the switched flow (i.e., the marginal cost). As the marginal cost is considered, other drivers’ route switching behavior and the resulting traffic conditions impact rational drivers’ route choice decisions. In the sense of marginal decision rule, drivers stop route switching when they perceive that the marginal benefit equals the marginal cost.

This paper introduces the marginal travel cost into the modeling of DTD traffic evolution, to enable consistency with the marginal decision rule that follows the economic principle that “rational people think at the margin” (Mankiw, 2012). It states that a rational individual evaluates his/her marginal utility, defined as the difference between the marginal benefit and the marginal cost, of each incremental decision. That is, both the marginal cost and the marginal benefit of each route switch need to be factored in the modeling of DTD traffic evolution. Hence, in this study we propose a conceptual shift in the definition of rationality by additionally introducing the notion of marginal cost rather than just the notion of marginal benefit considered in classical DTD models. The specification of the marginal cost in the proposed marginal utility DTD modeling framework may vary with the assumption on drivers' perception. This paper adopts the 1-step strategic thinking in behavioral game theory (Camerer, 2003) to derive an explicit formulation of the marginal cost, so that the analytical properties can be derived rigorously.

1.2 Literature review

The concept of rationality plays an important role in developing deterministic DTD models. Recently, Yang and Zhang (2009) focused on the behavioral rationality of deterministic DTD traffic evolution models. They note that most deterministic DTD evolution processes are rooted in deterministic user equilibrium (DUE), under which each driver cannot further reduce his/her travel cost by unilaterally switching to an alternative route (Wardrop, 1952). Their paper synthesizes the common characteristics of the proportional-switch adjustment process (Smith 1983; Smith, 1984; Smith and Wisten, 1995; Huang and Lam, 2002; Peeta and Yang, 2003), network tâtonnement adjustment (Friesz et al., 1994), projected dynamical system (Zhang and Nagurney, 1996; Nagurney and Zhang, 1997) and evolutionary traffic dynamics (Sandholm,
2001; Yang 2005). Yang and Zhang (2009) show that, under these traffic evolution processes, the route flow changes tend to reduce the aggregate travel cost given the previous day’s route costs. They label these processes as rational behavior adjustment processes (RBAP), whose stationary link flow patterns satisfy the DUE. These RBAP are all established on route flows.

In contrast to the route-based models summarized in Yang and Zhang (2009), some recent studies have focused on developing deterministic DTD models based on link flows. These link-based DTD models adopt the same concept of rationality as route-based DTD models. He et al. (2010) point out that route-flow based models suffer the inherent shortcomings of route non-uniqueness and route overlap. A link-based DTD model is then developed, where traffic evolution is characterized by link flow dynamics. The link-based DTD model has been applied to study the traffic evolution after the I-35W Bridge collapse in Minneapolis, Minnesota (He and Liu, 2012). Recently, three approximation models have been developed by Wang et al. (2015) to improve the model transferability of the link-based DTD model. Smith and Mounce (2011) propose another link-based DTD model, where link flows are determined by the flow splitting rates at each node. The adjustment of the link flow splitting rate enables incorporating node-level controls into the traffic evolution model. He et al. (2015) extend Smith and Mounce’s model by introducing a projection operator at the node level. Guo et al. (2013) extend the RBAP to the link level, and propose a discrete-time rational adjustment process. Guo et al. (2015) provide a general modeling framework for continuous-time link-based DTD models. They show that the existing link-based DTD models satisfy the continuous-time modeling framework. These link-based DTD models have the DUE flow as their stationary point.

A category of deterministic models that differs significantly from others is based on the boundedly rational (BR) choice behavior. In contrast to the rational behavior adjustment processes, the BR-DTD traffic evolution model assumes that drivers can take any route whose travel cost is within an “indifference band” of the shortest route cost. Thereby, its stationary link flows satisfy the boundedly rational user equilibrium (BRUE) instead of the DUE. Guo and Liu (2011) introduce boundedly rationality into modeling DTD traffic evolution. The BR model is used to explain the phenomenon that the traffic state is not restored after the reopening of I-35W Bridge. Using the BR traffic evolution model, Guo (2013) develops a sequential tolling strategy to guide the traffic flows to converge to a point in the untolled BRUE set which provides the best system performance without tolls. Wu et al. (2013) integrate the correlation of passenger flow
into a BR-based DTD model and use the model to study the passenger flow evolution in an urban railway network. It is worth noting that, similar to the RBAP, these BR-DTD models do not consider the route cost increases due to the route switching behavior.

The concept of rationality is critical to study the stability properties of DTD models. In the literature, a large body of research focuses on studying the stability and attraction domains of deterministic DTD traffic evolution. Based on the rationality assumption, Bie and Lo (2010) show that stationary points may not be attractive for a DTD traffic evolution process and the convergence property is sensitive to perturbation. The set of unstable states divides the whole space into different attraction domains. In this case, attraction domain becomes important to study the stability of DTD traffic evolution. Mounce and Carey (2011) discuss the stability issues, especially the convergence property in continuous-time space, for various route switching processes including pairwise switching, switching towards the least costly route, switching from above average to below average cost routes, and a projection type route switching process. Han and Du (2012) focus their study on the global stability of the link-based DTD model in He et al. (2010) and extend the stability property to a scenario with non-separable link cost functions. Cantarella (2013) analyzes the local and global stability issues when the design of Intelligent Transportation Systems is integrated with the DTD traffic evolution. All of these stability analyses are performed on the basis of driver route choice rationality.

The recent advances in DTD traffic evolution modeling enable its application to dynamic traffic control and operation: (i) Friesz et al. (2004), Yang and Szeto (2006), Yang et al. (2007), and Guo et al. (2013) apply deterministic models to DTD dynamic road pricing policies; (ii) Smith and Mounce (2011) apply the flow splitting rate model to DTD signal timing; (iii) Ye and Yang (2013) develop a continuous-time DTD model for the tradable credit pricing problem; and (iv) Cantarella et al. (2013) propose a class of DTD traffic evolution models to the design of transit operator strategies.

In summary, most existing deterministic DTD models, their stability analysis and applications rely on the assumption of driver rationality that is limited to the marginal benefit defined as the difference in perceived route costs. However, in a long-term planning context, drivers are able to perceive the cost sensitivity to link/route flows through their day-to-day travel experience as discussed in Kumar and Peeta (2015). Then, the marginal cost consisting of the travel cost increments due to driver route switching should be captured in a DTD model.
Theoretical properties, including properties of stationary point and stability, are worth investigating for DTD models accounting for the marginal cost.

To introduce the notion of marginal cost for the DTD model relies on the assumption that drivers are able to predict other drivers’ responses to the current traffic conditions, which is rooted in behavioral game theory (Camerer, 2003). The development of behavioral game theory is based on studies on fictitious play introduced by Brown (1951) and Robinson (1951), where each player makes a decision by taking other players’ responses into account. Behavioral game theory has been tested through a set of cognitive experiments (Costa-Gomes et al., 2001; Weizsäcker 2003; Costa-Gomes and Crawford, 2006) and applied in economics (Camerer et al. 2002; Crawford, 2003; Crawford and Iriberri, 2007). It can be used to develop more general, precise and theoretical models to describe how a non-cooperative game evolves toward equilibrium state based on weak assumptions of rationality, equilibrium and player’s self-interest (Camerer, et al., 2004). This paper applies the notion of strategic thinking in behavioral game theory to construct a new DTD model such that the marginal cost can be modeled in the context of route switching behavior.

1.3 Contributions and organization

This paper advances the modeling of DTD traffic evolution by making the following specific contributions. First, the marginal decision rule, a fundamental principle in economics, is introduced into a DTD traffic evolution modeling framework, where marginal cost is associated with the cost sensitivity to link/route flows. Second, the proposed DTD model is consistent with the behavior tendency of drivers being able to perceive the cost sensitivity to link/route flows based on long-term experience. Third, drivers’ strategic thinking is incorporated in their route switching decision-making process, which allows the leveraging of the well-developed behavioral game theory to model DTD traffic evolution. Finally, theoretical properties of a specific MU-DTD model are rigorously analyzed, including the invariance property, asymptotic stability, and the relationships with the rational behavioral adjustment process.

The remainder of the paper is organized as follows. The next section provides the preliminaries for establishing the MU-DTD traffic evolution model. Section 3 proposes a DTD traffic evolution modeling framework which factors the marginal cost in the link flow dynamics. Based on the modeling framework, section 4 constructs a specific MU-DTD model and provides
a rigorous analysis of its theoretical properties. Section 5 applies the proposed model to a test network to demonstrate the stability of the model and compare the evolution trajectories with the model where the marginal cost is not considered. The key findings and future research directions are summarized in Section 6.

2. Preliminaries

This section introduces the notation and assumptions associated with the formulation of DTD traffic evolution, lemmas used for establishing stability properties, a link-based DTD traffic evolution model upon which the MU-DTD model is constructed, and a definition of rational behavior adjustment process.

2.1 Notation and assumptions

This section provides the notations and assumptions that will be used in the study.

\( \mathcal{G}(N,E) \) a directed network \( \mathcal{G} \) with node set \( N \) and link set \( E \);

\( a \in E \) link index;

\( P_{rs} \) set of routes connecting origin-destination (OD) pair \( (r,s) \);

\( p \in P_{rs} \) route index;

\( f_p \) flow on route \( p \);

\( f \) vector of route flows, \( f = (f_p) \);

\( x_a \) flow on link \( a \);

\( x \) vector of link flows, \( x = (x_a) \);

\( \dot{x} \) link flow dynamics, \( \dot{x} = \frac{dx}{dt} = \left( \frac{dx_a}{dt} \right) \);

\( q_{rs} \) travel demand between OD pair \( (r,s) \);

\( q \) vector of OD demands \( q = (q_{rs}) \);

\( c_0^a \) free flow travel time on link \( a \);

\( c_a(x) \) travel time on link \( a \), as a function of link flows \( x \);

\( c \) vector of link travel times, \( c = (c_a) \);
\( y_a \)  
“target” flow on link \( a \);  

\( y \) vector of “target” link flows, \( y = (y_a) \);  

\( \delta_{ap} \) link-route index; if route \( p \) traverses link \( a \), \( \delta_{ap} = 1 \), otherwise, \( \delta_{ap} = 0 \);  

\( \delta \) link-route incidence matrix, \( \delta = (\delta_{ap}) \);  

\( \phi_{rs,p} \) OD-route index; if route \( p \) connects OD \( (r, s) \), \( \phi_{rs,p} = 1 \), otherwise \( \phi_{rs,p} = 0 \);  

\( \Phi \) OD-route incidence matrix, \( \Phi = (\phi_{rs,p}) \);  

\( I \) Identity matrix.

For a transportation network, the following relationships are straightforward: \( x = \delta f \), which represents the relationship between link flow and route flow; \( q = \Phi f \), which represents the relationship between route flow and travel demand. Denote \( \Omega \), the feasible link flow set, as: \( \Omega = \{ x | x = \delta f, \Phi f = q, f \geq 0 \} \), which is compact (i.e., closed and bounded) and convex. Other notation will be defined when first introduced.

**Assumption 1.** The link performance function \( c(x) \) is non-negative, continuous over the feasible set \( \Omega \), and strictly monotonic, i.e.,

\[
(x - \bar{x})^T (c(x) - c(\bar{x})) > 0, \quad \forall x, \bar{x} \in \Omega.
\]

This assumption is reasonable in the context of transportation planning and can be satisfied by the Bureau of Public Roads (BPR) and conical volume-delay functions (Spiess, 1990). This assumption indicates that the link travel cost is a non-decreasing function of link volume. Note that the link performance function \( c(x) \) can be either separable or non-separable for capturing link flow interactions. The following lemma is associated with Assumption 1.

**Lemma 1.** [Ortega and Rheinboldt (1970), Chapter 5.4] If the link performance function \( c(x) \) is monotonic, then its Jacobian matrix, \( \nabla c(x) \), is positive semi-definite; if the link performance function \( c(x) \) is strictly monotonic, then its Jacobian matrix, \( \nabla c(x) \), is positive definite.

The following assumption will be used in our analyses later.

**Assumption 2.** The link performance function \( c(x) \) is Lipschitz continuous on feasible set \( \Omega \) with Lipschitz constant \( L \), i.e.,
\[ \|c(x) - c(\bar{x})\| \leq L \|x - \bar{x}\|, \forall x, \bar{x} \in \Omega. \] (2)

2.2 Strategic thinking in behavioral game theory

The marginal cost formulation proposed in this study is rooted in behavioral game theory that emphasizes players’ cognitive limits, learning capability, and utility perception (Camerer, 2003). Through a large number of designed experiments, behavioral game theory recognizes that most players do think strategically when playing non-cooperative games. Here, strategic thinking means that players make a decision based on analysis of what others might do.

In behavioral game theory, players can be categorized by the number of steps of strategic thinking. This study adopts the concept of thinking steps defined in Camerer et al. (2004), where \( k \)-step players are using \( k \) steps of strategic thinking. In behavioral game theory, 0-step players do not think strategically at all. They make a decision based on observation of others’ decisions or randomize equally across all strategies in a one-shot game. By contrast, \( k \)-step players (\( k \geq 1 \)) are assumed to know the strategies of others who are using fewer steps of strategic thinking. However, these players may not realize that others are thinking as “hard” as they are or even harder (Camerer, 1990, 2003).

Placed in the context of the above discussion, most classical DTD traffic evolution models were built on the assumption that all drivers are using 0-step strategic thinking, namely they switch routes only based the observation of traffic conditions on previous days. Through the analysis of drivers’ route choice behavior after the collapse of the I-35W Mississippi River Bridge, He and Liu (2012) noted that drivers made route choice decisions based on the predicted traffic flow pattern after network disruption. However, in their DTD model, drivers predict the potential traffic congestion only once. In essence, drivers are 0-step players except for the day immediately after a network disruption event. Therefore, it remains unclear about how DTD traffic flow would evolve if drivers are not 0-step players. We seek to analyze this by introducing the marginal cost term into DTD traffic evolution modeling.

2.3 Link-based day-to-day traffic evolution model

In this study, the MU-DTD traffic evolution model will be constructed on the link-based DTD model proposed in He et al. (2010). Given two scalars \( \kappa > 0 \) and \( 0 < \beta < 1 \), the link-based DTD model is formulated as:
\[ \dot{x}(t) = \kappa \left[ y(t) - x(t) \right]. \]  \tag{3}

Vector \( y(t) \) represents a “target” flow pattern which solves the minimization problem:

\[ \min_{y \in \Omega} \beta c(x(t))^T y + (1 - \beta) D(x(t), y). \]  \tag{4}

The minimization problem (4) provides a behavioral explanation of route switching: drivers seek to minimize their travel cost, as presented by \( \min_{y \in \Omega} c(x(t))^T y \), while they would minimize unnecessary changes, as presented by \( \min_{y \in \Omega} D(x(t), y) \).

In the objective function, \( D(x, y) \) is a certain function that measures the difference between \( x \) and \( y \), which satisfies the following assumption.

**Assumption 3.** [Assumption B in Han and Du (2012)] \( D(x_1, x_2) \) is a non-negative differentiable function satisfying \( D(x_1, x_2) = 0 \) if and only if \( x_1 = x_2 \), and \( D(x_1, x_2) \) is strictly convex in \( x_1 \) for each \( x_2 \).

Under the above assumption, Han and Du (2012) show the following lemmas.

**Lemma 2.** [Lemma 1 in Han and Du (2012)] Let \( x(t, x^0) \) be a trajectory of link-based DTD model (3) with initial link flow pattern \( x(0) = x^0 \in \Omega \). It holds that \( x(t, x^0) \in \Omega \) for all \( t \).

**Lemma 3.** [Lemma 3 in Han and Du (2012)] If Assumption 3 holds, then \( x^* \) is the DUE link flow if and only if \( x^* \) solves minimization problem (4), i.e. \( x^* = y(x^*) \).

Lemma 2 is known as the invariance property; and Lemma 3 shows the equivalence of the stationary point of link-based DTD model (3) with the DUE state.

The choice of measure function \( D(x, y) \) affects the representation of the traffic evolution trajectory. He et al. (2010) propose that:

\[ D(x, y) = \sum_{a \in L} \int_{x_a}^{x_a} \left[ c_a(w) - c_a(x_a) \right] dw. \]  \tag{5}

However, this measure function relies on the assumption that the Jacobian of link cost functions must be symmetric. Han and Du (2012) investigate the effect of measure function \( D(x, y) \) under
the case of non-separable link cost functions with asymmetric Jacobian. They define the measure function $D(x, y)$ as:

$$D(x, y) = (x - y)^T B (x - y),$$

where $B$ is a symmetric positive definite matrix. If matrix $B$ is set to be an identity matrix, then $D(x, y) = \|x - y\|^2$ is the Euclidean distance between $x$ and $y$. If network topology changes, the Euclidean distance measure is not robust due to the “dummy node” effect (He et al., 2010). Since topology change is out of the scope of this study, $D(x, y) = \|x - y\|^2$ is used to accommodate the non-separable link cost functions.

Denote $Pr_{\Omega}(u)$ as the projection operator which projects vector $u$ onto convex set $\Omega$, i.e.:

$$Pr_{\Omega}(u) = \arg \min_{v \in \Omega} \|v - u\|.$$  

If $D(x, y) = \|x - y\|^2$, then the “target” flow pattern $y(t)$ in the DTD model (3) has a concise representation as follows.

**Lemma 4.** ([Theorem 10.2.3 in Facchinei and Pang (2003)] Let $D(x, y) = \|x - y\|^2$ in the minimization problem (4). Then, the unique solution of the minimization problem (4) is given by $y(t) = Pr_{\Omega}[x(t) - \varphi c(x(t))], \text{where } \varphi = \beta/2(1-\beta) > 0$.

2.4 Rational behavior adjustment process

In the literature, Zhang et al. (2001) propose three fundamental behavioral assumptions in drivers’ DTD route choice behavior. Based on the third behavioral assumption of Zhang et al. (2001), Yang and Zhang (2009) define a rational behavior adjustment process (RBAP) as one satisfying the following mathematical formulation:

$$\hat{f}(t) = \begin{cases} \epsilon & \text{if } \Gamma \neq \emptyset \\ 0 & \text{if } \Gamma = \emptyset \end{cases} \text{where } \Gamma = \left\{ u(t) : \sum_{p \in P} u_p(t) = 0, C(t)^T u(t) < 0 \right\}.$$  

where $\hat{f}(t)$ represents the route flow dynamics, $u(t)$ is the vector of infinitesimal changes in route flows, $C(t)$ represents route cost vector, and $\Gamma$ is the set of all feasible directions that reduce the aggregate travel cost based on the previous day’s route costs. Guo et al. (2015) extend
the RBAP to the link level, where route flow and route cost vectors are replaced by link flow and link cost vectors.

The definition of RBAP can be derived from a potential function. If the Jacobian of link cost function is symmetric, then there exists a potential function:

$$ U(t) = \sum_{w} \int_{0}^{c(t)} c(w) \, dw, $$

which is the objective function in the optimization formulation of DUE. A RBAP requires that the derivative of the aggregate travel cost w.r.t. time $t$ (in a day-to-day sense) satisfies:

$$ \dot{U}(t) = \frac{dU(t)}{dt} = \frac{\partial U(t)}{\partial x} \frac{dx}{dt} = c(x)^T \dot{x}(t) \leq 0. $$

where the “equal to” sign holds only when $x(t)$ is a stationary point, i.e., $\dot{x}(t) = 0$. Therefore, the potential function $U(t)$ can be regarded as an aggregate disutility function and a RBAP tends to reduce the aggregated disutility $U(t)$ at all $t$, unless the state is a stationary point. Based on (10), Theorem 9 in Han and Du (2012) also shows that the link-based DTD model (3) is a RBAP.

3. General framework of marginal utility day-to-day traffic evolution model

In this section, a new DTD traffic evolution modeling framework is developed. The marginal cost to be considered in drivers’ route switching decision will be defined first, followed by the mathematical formulation of the modeling framework as well as the invariance property of the proposed model.

3.1 Formulating marginal cost

Economists use the marginal decision rule to explain how people make decisions. The marginal decision rule states that rational people make their decisions by thinking at the margin (Mankiw, 2012). Here, “margin” is defined based on a person’s current decision. A small incremental adjustment of a person’s current decisions is called “marginal change”, which is made by comparing the marginal benefit and marginal cost.

The marginal decision rule is suitable for modeling DTD traffic flow evolution. In the context of DTD traffic flow evolution, DTD link flow changes result from drivers’ daily route choice decisions. Drivers’ route switching behavior from one day to another can be seen as the
“marginal change” in the marginal decision rule. Thereby, a DTD traffic evolution model describes the incremental adjustments of drivers’ route choice.

In the classical DTD models, the marginal benefit in the marginal decision rule is represented as the difference in a driver’s perceived route costs. However, the marginal cost due to the route switch has not been modeled. In a long-term planning context, drivers are able to perceive the link/route cost variations due to the flow fluctuations. If drivers can perceive the link/route cost-sensitivity through experience (or through advanced information systems), the marginal cost due to the route switching behavior can be measured and incorporated in deterministic DTD models. For a route with a higher marginal cost, namely, whose route cost would increase significantly with one unit of additional inflow, it is less attractive to drivers under the marginal decision rule. This study seeks to capture this route choice behavior by incorporating the marginal cost in deterministic DTD modeling.

In the context of DTD route choice, the marginal cost is the additional cost associated with changing route within a given time interval. Therefore, the marginal cost can be represented by:

$$\dot{c}(t) = \frac{dc(t)}{dt} = \frac{\partial c(t)}{\partial x} \cdot \frac{dx(t)}{dt} = \nabla c(x(t)) \cdot \dot{x}(t).$$

However, drivers cannot have perfect prediction of the traffic dynamics $\dot{x}(t)$ unless they are as sophisticated as $\infty$-step players (we will discuss it later). In this study, drivers are assumed to have perceived marginal costs $\hat{c}(t)$ based on predicted link flow dynamics $\hat{x}(t)$ that is defined as:

$$\hat{c}(t) = \nabla c(x(t)) \cdot \hat{x}(t), \quad (11)$$

where the perceived marginal cost $\hat{c}(t)$ is defined as the travel cost change due to the predicted route switch behavior and $\hat{x}(t)$ indicates the flow dynamics predicted by drivers.

In the proposed model, the marginal cost provides the capability to explain the behavioral tendency of the driver perceiving the cost-sensitivity to link/route flows. Kumar and Peeta (2015) seek to model this tendency by introducing the first derivative of route cost in determining the route switching rate in a deterministic DTD traffic evolution model. The proposed model is able to capture the same tendency because the marginal cost involves the slope of the link/route cost function, which represents cost-sensitivity to flow. In the context of the deterministic DTD
traffic evolution, drivers’ reluctance to use high sensitivity links/routes is reflected by a large value of the perceived marginal cost \( \tilde{c}(t) \).

Note that the perceived marginal cost (11) indicates the link (or route) cost change on the following day due to drivers’ response to the current traffic conditions. It emphasizes the cost induced by drivers’ route choice changes. This definition corresponds to the marginal cost defined in economics, where marginal cost represents the additional cost to produce one additional unit of good. By contrast, in transportation literature, the marginal cost is defined as the cost imposed on all users due to an additional vehicle in traffic (Mohring, 1976), which is mathematically formulated as \( c'(x) + \nabla c(x) \cdot \dot{x} \). In the perceived marginal cost (11), the term \( \nabla c(x(t)) \cdot \dot{x}(t) \) is the additional cost imposed to each individual driver given the switched flow \( \dot{x}(t) \). We will introduce this notion into a modeling framework for DTD traffic evolution in the next section.

3.2 Modeling framework

In general, a classical continuous-time DTD traffic evolution model without the consideration of marginal cost can be formulated as a dynamical system:

\[
\dot{x}(t) = F(x(t)),
\]

(12)

where \( F(x(t)) \) describes the link flow change rates at time \( t \). From a behavioral point of view, function \( F(x(t)) \) explains drivers’ routing strategy under an observed (realized) flow pattern \( x(t) \). Therefore, it can be regarded as the strategy of 0-step players. Function \( F(x(t)) \) can be specified as the link-based DTD model (3) or the proportional switch adjustment process proposed by Smith (1984).

To factor the marginal cost into the DTD modeling, one feasible way is to introduce a function associated with the marginal cost in the dynamical system (12). Suppose a function \( G(\tilde{c}(t)) \) is used to describe the impact of the marginal cost on the flow dynamics. Then, the dynamical system (12) is revised as:

\[
\dot{x}(t) = F(x(t)) + G(\tilde{c}(t)).
\]

(13)
Dynamical system (13) balances impacts from the marginal benefit characterized by \( F(x(t)) \) as well as the marginal cost characterized by \( G(\tilde{c}(t)) \). As formulation (13) involves both the marginal benefit and the marginal cost to characterize the DTD flow dynamics, we label the dynamical system (13) as marginal utility day-to-day (MU-DTD) model. Note that the proposed modeling framework is generic. It varies with the assumption on the perceived marginal cost, which may not rely on strategic thinking. Other notions, for example, the out-of-pocket monetary cost for receiving route travel time information, can be used to formulate the marginal cost in the MU-DTD modeling framework as well.

For simplicity, we specify \( G(\tilde{c}(t)) \) as a linear function of the perceived marginal cost \( \tilde{c}(t) \) in this study. Based on the definition of marginal cost (11),

\[
G(\tilde{c}(t)) = -\mu \tilde{c}(t) = -\mu \nabla c(x(t)) \cdot \dot{x}(t).
\]

The non-negative scalar \( \mu \) measures how much the drivers weigh the marginal cost in their route switching decisions. In the formulation, the perceived marginal cost \( \tilde{c}(t) \) is associated with a negative sign as it affects drivers’ route decisions negatively. Drivers would then perceive that their route switching benefits reduced more due to a larger value of marginal cost.

Dynamical system (13) can be specified as:

\[
\dot{x}(t) = F(x(t)) - \mu \nabla c(x(t)) \cdot \dot{x}(t).
\]

The behavioral explanation of equation (15) is that a link (or route) becomes less attractive if drivers predict that other drivers would switch to it on the next day. Note that equation (15) is formulated based on drivers’ prediction of flow dynamics \( \dot{x}(t) \). By adopting different assumptions on drivers’ strategic thinking on other drivers’ response to the current traffic conditions, different MU-DTD models can be formulated using equation (15).

Based on behavioral game theory, if drivers are 0-step players, namely, they make route choice decision based on the historical observations and do not predict other drivers’ responses to the current traffic conditions, then \( \dot{x}(t) = 0 \). The MU-DTD model becomes a DTD model without the consideration of marginal cost. If drivers are 1-step players, they predict that all other drivers are 0-step players (i.e., they make route choice based on DTD model (12) without the
marginal cost); then, \( \ddot{x}(t) = F(x(t)) \). Correspondingly, the MU-DTD model (15) can be represented as:

\[
\dot{x}(t) = F(x(t)) - \mu \nabla c(x(t)) F(x(t)) = \left[ I - \mu \nabla c(x(t)) \right] F(x(t)).
\] (16)

Further, if drivers are 2-step players, then they predict that other drivers are 1-step players. Therefore, each driver predicts that other drivers make route choice according to the DTD model (16), i.e., \( \ddot{x}(t) = \left[ I - \mu \nabla c(x(t)) \right] F(x(t)) \). Under this condition, the MU-DTD model (15) can be specified as:

\[
\dot{x}(t) = F(x(t)) - \mu \nabla c(x(t)) F(x(t)) + \mu^2 \left( \nabla c(x(t)) \right)^2 F(x(t)).
\]

Hence, the MU-DTD model (15) can be generalized under the assumption of \( k \)-step players as:

\[
\dot{x}(t) = \sum_{i=0}^{k} \left( -\mu \right)^i \left( \nabla c(x(t)) \right)^i F(x(t)).
\] (17)

For an extreme case that all drivers are \( \infty \)-step players, though it is not realistic, the MU-DTD model can be formulated as:

\[
\dot{x}(t) = \sum_{i=0}^{\infty} \left( -\mu \right)^i \left( \nabla c(x(t)) \right)^i F(x(t)) = \left( I + \mu \nabla c(x(t)) \right)^{-1} F(x(t)).
\] (18)

Note that based on Assumption 1 and Lemma 1, \( \nabla c(x(t)) \) is a positive definite matrix, and so is matrix \( I + \mu \nabla c(x(t)) \). Therefore, \( I + \mu \nabla c(x(t)) \) is invertible. The above equation implies that:

\[
\dot{x}(t) = F(x(t)) - \mu \nabla c(x(t)) \dot{x}(t).
\] (19)

In this case, the marginal cost \( G(\dot{x}(t)) = -\mu \nabla c(x(t)) \dot{x}(t) = -\mu \dot{c}(t) \). Note that in dynamical system (19), both sides of the equation relate to the traffic flow dynamics \( \dot{x}(t) \).

In behavioral game theory, the 1-step decision rule has been studied extensively. Camerer (1990) analyzed the main properties of the 1-step decision rule. Nagel (1995), Stahl and Wilson (1995), Costa-Gomes and Crawford (2006) and Weizsäcker (2003) designed a large number of experiments and found that 1-step rules are most commonly adopted by players. Note that these experiments were designed using specific games (p-Beauty contest, tic-tac-toe, etc.) and tested on relatively small number of subjects in contrast to the number of drivers in a transportation network. However, the 1-step strategic thinking provides an appropriate framework to model how drivers predict others’ choices when making route decisions. Therefore, the rest of this
paper only focuses on the case that all drivers are 1-step players, i.e., the dynamical system (16). It is summarized as the following assumption.

**Assumption 4.** Drivers are homogeneous and 1-step strategic thinkers, such that their perceived marginal cost is \( \tilde{c}(t) = \nabla c(x(t)) F'(x(t)) \).

This assumption is reasonable when the perfect information assumption is valid as it implies knowledge of the actions of other drivers. Further, insights from a previous study by He and Liu (2012) using the traffic flow evolution data collected after the collapse of the I-35W Mississippi River Bridge suggest this assumption. This is also reinforced by research in the domain of game theory that motivated the development of behavioral game theory (Camerer, 2003) where players are able to predict other players’ future moves. Finally, such information can also be assumed to be available through advanced information provision systems.

The stationary point of dynamical system (16) can be analyzed. Based on Assumption 1 and Lemma 1, \( \nabla c(x(t)) \) is a positive definite matrix for all \( x(t) \in \Omega \). Denote \( \rho(x(t)) \) as the maximum eigenvalue (i.e. spectral radius) of \( \nabla c(x(t)) \). Note that based on Assumption 2, \( c(x) \) is Lipschitz continuous on feasible set \( \Omega \), thus each element in \( \nabla c(x(t)) \) is bounded for all \( x(t) \in \Omega \). Therefore, \( \rho(x(t)) \) is bounded. Denote \( \bar{\rho} = \max_{x(t) \in \Omega} \rho(x(t)) \). If \( \mu < 1/\bar{\rho} \), then the maximum eigenvalue of matrix \( \mu \nabla c(x(t)) \) is smaller than one for all \( x(t) \in \Omega \). Matrix \( I - \mu \nabla c(x(t)) \) is invertible. The following theorem establishes the relationship between dynamical system (16) and its classical counterpart, i.e., the DTD model (12).

**Theorem 1.** Suppose that Assumption 1 holds. Let \( \mu \) be a scalar such that \( \mu < 1/\bar{\rho} \) in dynamical system (16). A link flow pattern \( x^* \) is a stationary point of the DTD model (12) if and only if \( x^* \) is a stationary point of MU-DTD model (16).

**Proof.** Due to Assumption 1 and Lemma 1, \( \rho(x(t)) > 0 \) for all \( x(t) \in \Omega \). As \( \Omega \) is a bounded closed set, there exists \( \bar{\rho} = \max_{x(t) \in \Omega} \rho(x(t)) > 0 \). Then, \( I - \mu \nabla c(x(t)) \) is invertible as \( \mu < 1/\bar{\rho} \).

**Sufficiency:** If \( x^* \) is a stationary point of dynamical system (16), then:

\[
\dot{x}^*(t) = \left[I - \mu \nabla c(x^*(t))\right] F'(x^*(t)) = 0. \tag{20}
\]
As matrix $\mathbf{I} - \mu \nabla c(\mathbf{x}^*(t))$ is invertible, the above equation implies that $F(\mathbf{x}^*(t)) = 0$. Thus, $\mathbf{x}^*$ is a stationary point of the DTD model (12).

**Necessity:** If $\mathbf{x}^*$ is a stationary point of DTD model (12), then $F(\mathbf{x}^*(t)) = 0$. Thus, 

$$\left[ \mathbf{I} - \mu \nabla c(\mathbf{x}^*(t)) \right] F(\mathbf{x}^*(t)) = 0,$$

and $\mathbf{x}^*$ is a stationary point of dynamical system (16). ■

**Remark 1.** Theorem 1 is developed based on the specification of linear function (14). If $\mu$ is large or function $G\left(\mathbf{\tilde{e}}(t)\right)$ is specified using other representations, e.g., including a constant term to represent additional costs, then dynamical system (16) may have a stationary point that differs from the DTD model (12).

**Remark 2.** Although dynamical system (16) has a simple representation, it does not guarantee the invariance property as shown in Lemma 2. Particularly, assume that an initial link flow $\mathbf{x}^0$ is in the feasible set $\Omega$. Then, the trajectory of $\mathbf{x}\left(\mathbf{x}^0, t\right)$ characterized by dynamical system (16) may not stay in the feasible set $\Omega$ for all $t$, due to the existence of the Jacobian matrix $\nabla c(\mathbf{x})$.

We will address this issue in the next section.

3.3 Maintaining the invariance property

To ensure the invariance property is a challenging task in the modeling of DTD traffic evolution. In the literature, an additional step is commonly introduced to the model. For example, network tâtonnement adjustment (Friesz et al., 1994) and projected dynamical system (Nagurney and Zhang, 1997) introduce a projection operator in the model formulation; the link-based DTD model (He et al., 2010) requires solving a minimization sub-problem; the flow splitting rate model (Smith and Mounce, 2011) requires a post-assignment link flow adjustment process.

In this study, we follow the network tâtonnement model (Friesz et al., 1994) to ensure the invariance property of dynamical system (16). The network tâtonnement model applies the projection operator defined by (7) to avoid the violation of flow conservation. Denote

$$\mathbf{d}(\mathbf{x}(t)) = -F(\mathbf{x}(t)) + \mu \nabla c(\mathbf{x}(t)) F(\mathbf{x}(t)).$$

(21)

Then $\mathbf{d}(\mathbf{x}(t))$ can be regarded as the marginal disutility of route switching. Denote scalar $\tau > 0$. Thereby, the DTD model can be presented as:
As the marginal disutility contains a marginal cost term in dynamical system (22), it is a MU-DTD model. Note that the definition of marginal disutility $d(x(t))$ can be revised according to the assumption on drivers’ strategic thinking, for example, equation (17) or (18). To illustrate the application of dynamical system (22), the next section specifies function $F(x(t))$ as the link-based DTD model (3) and analyzes the properties of the resulting specific MU-DTD model.

4. A specific MU-DTD model

4.1 Formulation

The modeling framework (22) can be used to develop different MU-DTD models by simply replacing $F(x(t))$ with a specific continuous-time DTD model. To illustrate the application of the framework (22), the link-based DTD model (3) is used in formulation (12); that is $F(x(t)) = \kappa[y(t) - x(t)]$. Thereby, the dynamical system (16) can be specified as:

$$\dot{x}(t) = \kappa[I - \mu \nabla c(x(t))] [y(t) - x(t)].$$

(23)

where $y(t)$ solves the minimization problem (4). The following theorem establishes the relationship between a stationary point of (23) and the DUE.

**Theorem 2.** Suppose that Assumption 1 holds and $\mu$ satisfies $\mu < 1/\bar{\rho}$ in the MU-DTD model (23). A link flow pattern $x^*$ is a stationary point of dynamical system (23) if and only if $x^*$ is the DUE link flow.

**Proof.** Based on Theorem 1, a link flow pattern $x^*$ is a stationary point of dynamical system (23) if and only if $x^*$ is a stationary point of the link-based DTD model (3), i.e., $\dot{x}^*(t) = \kappa[y^*(t) - x^*(t)] = 0$. Due to Lemma 3, $x^*$ is a stationary point of the link-based DTD model (3), if and only if $x^*$ is the DUE link flow. This completes the proof. □

According to equation (22), a MU-DTD model is formulated as:

$$\dot{x}(t) = \tau \left\{ p_{\alpha} \right\} [x(t) - \gamma d(x(t))] - x(t).$$

(22)

According to (21), function $d(x(t))$ is defined as:
where \( y(t) \) is the “target” flow pattern in the link-based DTD model (3). Based on Lemma 4, if \( x(t) - \varphi c(x(t)) \in \Omega \), then \( x(t) - y(t) = \varphi c(x(t)) \). According to equation (11), the second term in the right-hand-side of equation (25) represents the perceived marginal cost \( \hat{c}(x(t)) \) under the assumption of 1-step player. Then, the definition of \( d(x(t)) \) is equivalent to \( d(x(t)) = \kappa \varphi c(x(t)) + \kappa \mu \hat{c}(x(t)) \). It shows that the marginal disutility \( d(x(t)) \) can be seen as a weighted average of the average cost \( c(x(t)) \) and the perceived marginal cost \( \hat{c}(x(t)) \).

For simplicity, denote \( z(t) = \Pr_{\Omega}[x(t) - \gamma d(x(t))] \). Base on the definition of projection operator (7), link flow pattern \( z(t) \) is an optimizer of the minimization problem:

\[
\min_{z \in \Omega} \eta d(x(t))^T z(t) + (1-\eta)\|x(t) - z(t)\|^2,
\]  

where \( 0 < \eta < 1 \) is a positive scalar satisfying \( \eta = 2\gamma/(1+2\gamma) \). The optimization problem (26) can be viewed as a weighted average of two sub-problems:

\[
\min_{z \in \Omega} d(x(t))^T z(t),
\]  

and

\[
\min_{z \in \Omega} \|x(t) - z(t)\|^2.
\]  

Sub-problem (27) solves for a feasible link flow pattern that minimizes the perceived marginal disutility \( d(x(t)) \) for each OD pair. Sub-problem (28) captures drivers’ reluctance to change their current routes as discussed in He et al. (2010). The uniqueness of the optimization problem (26) is guaranteed due to the strict convexity of Euclidean distance \( \|x(t) - z(t)\|^2 \) given by Assumption 3. Due to Lemma 4, link flow pattern \( z(t) \) is the unique solution to the minimization problem (26).

The link-based MU-DTD model (23) can be modified as:

\[
\dot{x}(t) = t(x(t) - x(t)),
\]
where \( \mathbf{z}(t) \in \Omega \) solves the minimization problem (26). Feasibility of \( \mathbf{z}(t) \) helps to maintain the invariance property of the dynamical system (29).

In optimization sub-problem (26), computing the marginal disutility \( \mathbf{d}(\mathbf{x}(t)) \) requires the calculation of the Jacobian \( \nabla \mathbf{c}(\mathbf{x}(t)) \). An approximation of the marginal cost can provide a simpler representation of the marginal disutility \( \mathbf{d}(\mathbf{x}(t)) \). Let \( \Delta t = 1 \) such that \( \dot{\mathbf{x}} = \Delta \mathbf{x}/\Delta t = \Delta \mathbf{x} \). Applying a Taylor series expansion to the link cost vector:

\[
\mathbf{c}(\mathbf{x} + \Delta \mathbf{x}) = \mathbf{c}(\mathbf{x}) + \nabla \mathbf{c}(\mathbf{x}) \Delta \mathbf{x} + o(\Delta \mathbf{x}).
\]  

The marginal cost can be approximated by:

\[
\nabla \mathbf{c}(\mathbf{x}(t)) \Delta \mathbf{x}(t) = \mathbf{c}(\mathbf{x}(t) + \Delta \mathbf{x}(t)) - \mathbf{c}(\mathbf{x}(t)) + o(\Delta \mathbf{x}).
\]  

Based on the link-based DTD model (3), let \( \mathbf{c}(\mathbf{x}(t) + \Delta \mathbf{x}(t)) = \mathbf{c}(\mathbf{y}(t)) \) in this study; then the marginal disutility of route switching is redefined as:

\[
\mathbf{d}(\mathbf{x}(t)) \approx \kappa [\mathbf{x}(t) - \mathbf{y}(t)] + \kappa \mu [\mathbf{c}(\mathbf{y}(t)) - \mathbf{c}(\mathbf{x}(t))].
\]  

which provides a simpler representation of the marginal disutility \( \mathbf{d}(\mathbf{x}(t)) \) for the proposed MU-DTD model (29).

Note that the representation of the marginal disutility function (32) can be seen as a second order approximation in the Douglas-Rachford splitting method for solving partial differential equations (Douglas and Rachford, 1956), convex programming (Tseng, P., 1991), and variational inequality (He, 1997). Good mathematical properties exist for this representation as discussed in the next section.

4.2 Properties of the MU-DTD model

This section presents the main properties of the proposed MU-DTD model (29). The invariance property and the property of the stationary state of the proposed model are first analyzed, followed by the analysis of the asymptotic stability in the sense of Lyapunov. Finally, the rational behavior adjustment process is established based on the marginal decision rule.
4.2.1 Invariance property

Based on its explicit representation, the invariance property of the proposed MU-DTD model (29) can be developed as follows.

**Theorem 3.** Let initial link flow $x(0) = x^0$ be in the feasible set $\Omega$. The link flow trajectory $x(t, x^0)$ characterized by the MU-DTD model (29) satisfies that $x(t, x^0) \in \Omega$ for all $t$.

**Proof.** Given link flow vector $x(t)$, $z(t)$ is the unique solution to minimization problem (26). Thus, $z(t) \in \Omega$ for all $t$. Due to Lemma 2, it is straightforward that $x(t, x^0) \in \Omega$ for all $t$.  

4.2.2 Property of the stationary state

The following lemma is needed for further analysis of the stationary state of the MU-DTD model (29).

**Lemma 5.** Suppose that Assumption 1 holds. For a positive constant $0 < \nu < 1$, there exists a positive $\alpha_{\infty} > 0$, such that:

$$\alpha_{\infty} \left[ y(x) - x \right]^T [c(y(x)) - c(x)] \leq (1 - \nu) \| y(x) - x \|^2 $$

holds for all $x \in \Omega$.

**Proof.** Note $y(x) \in \Omega$. As shown in Theorem 1, there exists a $\bar{\rho} = \max_{y \in \Omega} \rho(y) > 0$ where $\rho(y)$ represents the maximum eigenvalue of $\nabla c(y(x))$. Due to Assumption 1 and Lemma 1, $\nabla c(y(x))$ is a positive definite matrix for all $y(x) \in \Omega$. Therefore,

$$\nabla c(y(x)) = R^T(y(x)) \Lambda(y(x)) R(y(x)),$$

where $R(y(x))$ is an orthogonal matrix, and $\Lambda(y(x))$ is a diagonal matrix with the eigenvalues of $\nabla c(y(x))$ as its diagonal entries. As eigenvalues of $\nabla c(y(x))$ are smaller or equal to $\bar{\rho}$ for all $y(x)$, we have the following inequality:

$$\left[ y(x) - x \right]^T \nabla c(y(x)) [y(x) - c(x)] \leq \bar{\rho} \left[ y(x) - x \right]^T R^T(y(x)) R(y(x)) [y(x) - c(x)]$$

$$= \bar{\rho} \| y(x) - x \|^2. \quad (35)$$
Note that $R^T(y(x))R(y(x)) = I$ as $R(y(x))$ is orthogonal. Applying Taylor series expansion (30) yields
\[
[y(x) - x]^T[\nabla c(y(x))(y(x) - x) + o(y(x) - x)]
= [y(x) - x]^T \nabla c(y(x))(y(x) - x) + o(y(x) - x)
\leq (\bar{\rho} + \varepsilon)\|y(x) - x\|^2
\] (36)
where $\varepsilon$ is a small positive constant. For any given $0 < \nu < 1$, define $\alpha_{\max} = (1 - \nu)/(\bar{\rho} + \varepsilon)$. Inequality (33) is a direct result of inequality (36).

**Corollary 1.** Suppose that Assumption 2 holds. Then, for a positive constant $0 < \nu < 1$, there exists a positive $\alpha_{\max} > 0$, such that inequality (33) holds for all $x \in \Omega$.

**Proof.** Due to Assumption 2, $\|c(x) - c(y(x))\| \leq L \|x - y(x)\|$ for all $x \in \Omega$, where $L$ is Lipschitz constant. For any given $0 < \nu < 1$, let $\alpha_{\max} \leq (1 - \nu)/L$, then:
\[
\alpha_{\max} \|c(x) - c(y(x))\| \leq (1 - \nu)\|x - y(x)\|, \forall x \in \Omega.
\] Inequality (33) holds readily due to Cauchy-Schwarz inequality.

Given Lemma 5, we can show the relationship between a stationary point of the proposed MU-DTD model (29) and the DUE link flow as follows.

**Theorem 4.** Let parameter $\mu$ in the marginal disutility function (32) satisfy $\mu \leq \alpha_{\max}$, where $\alpha_{\max} = (1 - \nu)/(\bar{\rho} + \varepsilon)$ for given $0 < \nu < 1$ and $\varepsilon > 0$. And, suppose that Assumptions 1 and 2 hold. Then, a link flow pattern $x^*$ is a stationary point of the MU-DTD model (29) if and only if $x^*$ is the DUE link flow.

**Proof.** **Sufficiency:** Let $x^*$ be the DUE link flow. Based on Lemma 3, $x^*$ solves the minimization problem (4), i.e., $x^* = y(x^*)$. Thus, the marginal disutility defined by equation (32) is equal to zero, i.e., $d(x^*) = 0$. Note that $z$ is the solution to the optimization problem (26), i.e.,
\[
z = \arg \min_{z \in \mathcal{Z}} \eta d(x^*)^T z + (1 - \eta)\|x^* - z\|^2 = \arg \min_{z \in \mathcal{Z}} \|x^* - z\|^2.
\] Thus $z = x^*$ and $x^* = 0$, i.e., $x^*$ is a stationary point of the MU-DTD model (29).
Necessity: Let \( \mathbf{x}^* \) be a stationary point of the MU-DTD model (29). Thus, \( \mathbf{z} = \mathbf{x}^* \) and \( \| \mathbf{x}^* - \mathbf{z} \|^2 = 0 \). As \( \mathbf{z} = \mathbf{x}^* \) is the minimizer of optimization problem (26), then:

\[
\mathbf{x}^* = \arg \min_{\mathbf{x} \in \Omega} \mathbf{z}^T \mathbf{d} \left( \mathbf{x}^* \right).
\]

Therefore, \( \mathbf{x}^T \mathbf{d} \left( \mathbf{x}^* \right) \leq \mathbf{\bar{x}}^T \mathbf{d} \left( \mathbf{x}^* \right) \) for any \( \mathbf{\bar{x}} \in \Omega \), i.e.:

\[
(\mathbf{\bar{x}} - \mathbf{x}^*)^T \mathbf{d} \left( \mathbf{x}^* \right) = (\mathbf{\bar{x}} - \mathbf{x}^*)^T \left\{ \kappa \left[ \mathbf{x}^* - \mathbf{y} \left( \mathbf{x}^* \right) \right] + \mu \left[ \mathbf{c} \left( \mathbf{y} \left( \mathbf{x}^* \right) \right) - \mathbf{c} \left( \mathbf{x}^* \right) \right] \right\} \geq 0, \quad \forall \mathbf{\bar{x}} \in \Omega. \tag{37}
\]

Let \( \mathbf{\bar{x}} = \mathbf{y} \left( \mathbf{x}^* \right) \) in (37). Lemma 5 is used to derive the following inequalities.

\[
0 \leq (\mathbf{y} \left( \mathbf{x}^* \right) - \mathbf{x}^*)^T \mathbf{d} \left( \mathbf{x}^* \right) = -\kappa \| \mathbf{x}^* - \mathbf{y} \left( \mathbf{x}^* \right) \|^2 + \kappa \mu \left[ \mathbf{y} \left( \mathbf{x}^* \right) - \mathbf{x}^* \right]^T \left[ \mathbf{c} \left( \mathbf{y} \left( \mathbf{x}^* \right) \right) - \mathbf{c} \left( \mathbf{x}^* \right) \right] \\
\leq -\kappa \| \mathbf{x}^* - \mathbf{y} \left( \mathbf{x}^* \right) \|^2 + \kappa \alpha_{\text{max}} \left[ \mathbf{y} \left( \mathbf{x}^* \right) - \mathbf{x}^* \right]^T \left[ \mathbf{c} \left( \mathbf{y} \left( \mathbf{x}^* \right) \right) - \mathbf{c} \left( \mathbf{x}^* \right) \right] \\
\leq -\kappa \| \mathbf{x}^* - \mathbf{y} \left( \mathbf{x}^* \right) \|^2 + \kappa (1 - \nu) \| \mathbf{x}^* - \mathbf{y} \left( \mathbf{x}^* \right) \|^2 \\
= -\kappa \nu \| \mathbf{x}^* - \mathbf{y} \left( \mathbf{x}^* \right) \|^2 \leq 0.
\]

The above inequalities imply \( \mathbf{x}^* - \mathbf{y} \left( \mathbf{x}^* \right) = 0 \). Due to Lemma 3, \( \mathbf{x}^* \) is a DUE link flow. \( \blacksquare \)

**Corollary 2.** Let parameter \( \mu \) in the marginal disutility function (32) satisfy \( \mu \leq \alpha_{\text{max}} = \left( 1 - \nu \right) / \left( \overline{\rho} + \varepsilon \right) \) for given \( 0 < \nu < 1 \) and \( \varepsilon > 0 \), and suppose Assumptions 1 and 2 hold. Then, a link flow pattern \( \mathbf{x}^* \) is a stationary point of the MU-DTD model (29) if and only if \( \mathbf{x}^* \) solves the fixed point problem:

\[
\mathbf{d} \left( \mathbf{x}^* \right) = 0. \tag{38}
\]

**Proof.** This corollary holds readily due to the definition of disutility function (32), Lemma 3, and Theorem 4. \( \blacksquare \)

**Remark 3.** From Theorem 4, whether a stationary point of the MU-DTD model (29) is a DUE flow depends on the value of parameter \( \mu \) in the system, namely, how much drivers weigh the perceived marginal cost \( \mathbf{v} \mathbf{c} \left( \mathbf{x} \left( t \right) \right) \cdot \dot{\mathbf{x}} \left( t \right) \) in the marginal disutility function. If drivers are sensitive to the marginal cost, i.e., \( \mu > \alpha_{\text{max}} \), then a stationary point of the proposed MU-DTD model (29) may not be a DUE flow.
Remark 4. Corollary 2 illustrates the marginal decision rule in economics, where consumers stop consuming goods when perceived marginal benefits equal perceived marginal costs. In the context of DTD route decision-making, drivers stop changing their routes when the marginal utility of route switching $d(x)$ equals zero.

4.2.3 Asymptotic stability

In the literature, Fukushima (1992) established a connection between the gap minimization problem and asymmetric variational inequalities, where an objective function similar to the objective function in (4) is used to measure the distance between an intermediate solution and the optimal solution set. The same insight is applied here. Due to the definition of marginal disutility (32) and Corollary 2, the MU-DTD model (29) implicitly provides a measurement of the distance between the current link flow pattern and the stationary link flow set of the proposed DTD link flow dynamical system. The present study applies this relationship in the analysis of system stability.

Our stability analysis starts with two definitions of stability, similar to those defined in Nagurney and Zhang (1997) and Bie and Lo (2010).

Definition 1. (Stability in the sense of Lyapunov) The MU-DTD model (29) is stable if for every initial link flow pattern $x(0)=x^0$ and a stationary link flow, $x^*$, the Euclidean distance $V(x(t,x^0),x^*)=\frac{1}{2}\|x(t,x^0)-x^*\|^2$ is a monotone non-increasing function of time $t$.

Definition 2. (Asymptotic stability) The MU-DTD model (29) is asymptotically stable if it is stable, and for every initial link flow pattern $x(0)=x^0$ there exists a stationary link flow, $x^*$, such that $\lim_{t \to \infty} x(x^0,t)=x^*$.

For simplicity, denote $x(t)=x(x^0,t)$ for any given $x^0$. Note that function $V(x(t),x^*)$ is a Lyapunov function.

Lemma 6. Let $x^*$ be a fixed point of the MU-DTD model (29) and $x(t)$ is a trajectory determined by the model. Suppose Assumptions 1 and 2 hold and parameter $\mu$ in the marginal
disutility function satisfies \( \mu \leq \alpha_{\text{max}} = (1 - \nu)/(\rho + \varepsilon) \) for given \( 0 < \nu < 1 \) and \( \varepsilon > 0 \). Then, inequality:

\[
(x(t) - x^*)^T[x(t) - z(t)] \geq \left[ \nu - \frac{\gamma^2(1 + \mu^2L^2)}{4} \right] \|x(t) - y(t)\|^2
\]

holds for all time \( t \).

**Proof.** See Appendix.

The asymptotic stability for the MU-DTD model (29) can now be established.

**Theorem 5.** Let \( x^* \) be a stationary point of the MU-DTD model (29) and \( x(t) \) be a trajectory determined by the model. Suppose that Assumptions 1 and 2 hold and parameter \( \mu \) in the marginal disutility function (32) satisfies \( \mu \leq \alpha_{\text{max}} = (1 - \nu)/(\rho + \varepsilon) \) for given \( 0 < \nu < 1 \) and \( \varepsilon > 0 \). If parameter \( \gamma \) in the optimization problem (24) satisfies:

\[
\gamma < \sqrt{\frac{4\nu}{1 + \mu^2L^2}},
\]

then, the MU-DTD model (29) is asymptotically stable.

**Proof.** The property that the MU-DTD model (29) is stable will be proved first. Take the derivative of the Euclidean distance \( V(x(t), x^*) = \frac{1}{2} \|x(t) - x^*\|^2 \) w.r.t. time \( t \):

\[
\dot{V}(x(t), x^*) = \frac{dV}{dt} = \frac{dV}{dx} \frac{dx}{dt} = \kappa (x(t) - x^*)^T[z(t) - x(t)].
\]

Due to Lemma 6,

\[
\dot{V}(x(t), x^*) \leq -\kappa \left[ \nu - \frac{\gamma^2(1 + \mu^2L^2)}{4} \right] \|x(t) - y(t)\|^2 \leq 0.
\]

The above inequality implies that the distance function \( V(x(t), x^*) \) is monotonically non-increasing w.r.t. time \( t \). By Definition 1, the MU-DTD model (29) is stable.

Next, the convergence of the trajectory determined by the MU-DTD model (29) will be proved. Because of (41), the distance function \( V(t) = V(x(t), x^*) \) is bounded and non-negative for all time \( t \). It must converge to a certain value \( \zeta \), i.e.:
\[ \lim_{t \to \infty} V(t) = \zeta. \] (42)

If \( \zeta > 0 \), then there must exist a sequence \( \{t_n\} \), \( t_n \to \infty \), as \( n \to \infty \), such that:
\[ \lim_{n \to \infty} \dot{V}(t_n) = 0. \] (43)

If this statement is false, then there exists a \( \psi > 0 \) and a large enough \( T > 0 \), such that
\[ \dot{V}(t) < -\psi, \quad \forall t > T, \]
which contradicts the non-negativity of \( V(t) \). Therefore, the convergence (43) holds.

The limit (42) implies that the sequence \( \{x(t_n)\} \) is bounded. Therefore, there exists a subsequence \( \{x(t_{n_j})\} \) such that:
\[ \lim_{j \to \infty} x(t_{n_j}) = \hat{x}. \] (44)

Apply the definition of \( V(t) \) to the limit:
\[ \lim_{j \to \infty} \frac{1}{2} \|x(t_{n_j}) - x^*\|^2 = \frac{1}{2} \|\hat{x} - x^*\|^2 = \zeta > 0. \] (45)

Hence \( \hat{x} \neq x^* \). Substituting \( x(t) \) by \( x(t_{n_j}) \) in (41) yields:
\[ \dot{V}(x(t_{n_j}), x^*) \leq - \left[ V - \frac{\gamma^2 (1 + \alpha^2 L^2)}{4} \right] \|x(t_{n_j}) - y(t_{n_j})\|^2 \leq 0. \] (46)

Due to (43), the left-hand-side of (46) converges to zero as \( j \to \infty \). Condition (40) implies that \( \left[ V - \gamma^2 (1 + \alpha^2 L^2)/4 \right] > 0 \). Equations (44) and (46) imply the limit:
\[ \lim_{j \to \infty} \|x(t_{n_j}) - y(t_{n_j})\|^2 = \|\hat{x} - y(\hat{x})\|^2 = 0. \] (47)

Hence \( \hat{x} = y(\hat{x}) \). Due to Lemma 3, \( \hat{x} = x^* \) which contradicts the earlier result from (45). This contradiction implies that \( \zeta = 0 \). According to (42),
\[ \lim_{t \to \infty} x(t) = x^*. \] (48)

By Definition 2, the MU-DTD model (29) is asymptotically stable. \( \blacksquare \)

**Remark 5.** Theorem 5 provides a sufficient condition (40) for the asymptotic stability of the MU-DTD model (29). Essentially, it requires \( \eta \) in the optimization problem (26) to be small
enough such that inequality (40) holds. However, due to the difficulty of estimating Lipschitz constant $L$ and the maximum eigenvalue $\bar{\rho}$ to determine $\alpha_{\text{max}} = (1-\nu)/(\bar{\rho} + \varepsilon)$, this sufficiency condition is generally difficult to verify.

**Remark 6.** In the proof of Theorem 5, inequality (41) shows that the larger the parameter $\kappa$, the smaller the negative value of $\dot{V}(x(t), x^*)$. Thereby, when condition (40) is satisfied, the MU-DTD model (29) stabilizes more rapidly with a larger $\kappa$, although the value of $\kappa$ does not impact the proof of asymptotic stability.

**4.2.4 Rational behavior adjustment process in the sense of the marginal decision rule**

The concept of RBAP can be shifted to the one based on the marginal decision rule, which is defined as follows.

**Definition 3 (Rationality in the sense of the marginal decision rule)** An adjustment process $\dot{x}(t)$ is called rational in the sense of the marginal decision rule, if it satisfies:

$$d(t)^T \dot{x}(t) \leq 0, \forall t,$$

and the “equal to” sign holds only at stationary points.

Note that the average cost $c(x)$ in the RBAP (10) is now replaced by the marginal disutility $d(x)$. Based on Definition 3, we have the following theorem.

**Theorem 6.** Let $x(t)$ be a trajectory determined by the MU-DTD model (29). Suppose that Assumptions 1 and 2 hold and parameter $\mu$ in the marginal disutility function (32) satisfies $\mu < \alpha_{\text{max}} = (1-\nu)/(\bar{\rho} + \varepsilon)$ for given $0 < \nu < 1$ and $\varepsilon > 0$. Then, the MU-DTD model (29) is rational in the sense of the marginal decision rule.

**Proof.** At any time $t$, $z(t)$ is a minimizer of the optimization problem (26), such that:

$$\eta d(x(t))^T z(t) + (1-\eta) \| x(t) - z(t) \|^2 \leq \eta d(x(t))^T x(t) + (1-\eta) \| x(t) - z(t) \|^2.$$

The above inequality implies that:

$$d(x(t))^T [z(t) - x(t)] \leq -(1-\eta)/\eta \| x(t) - z(t) \|^2 \leq 0.$$
The MU-DTD model (29) is defined as $\dot{x}(t) = \tau [z(t) - x(t)]$ with $\tau > 0$ and $0 < \eta < 1$. Then, inequality (49) holds for all time $t$.

For any $x$ that is not a stationary point of the MU-DTD model (29), based on Lemma 5, we have:

$$
(y(x) - x)^T d(x) = -\kappa \|x - y(x)\|^2 + \kappa \mu [y(x) - x]^T [c(y(x)) - c(x)] \\
\leq -\kappa \|x - y(x)\|^2 + \kappa \alpha_{\text{max}} [y(x) - x]^T [c(y(x)) - c(x)] \\
\leq -\kappa \|x^* - y(x)\|^2 + \kappa (1 - \nu) \|x - y(x)\|^2 \\
= -\kappa \nu \|x^* - y(x)\|^2 < 0.
$$

Therefore, the “equal to” sign of inequality (49) holds only when $x(t)$ is a stationary point and the link flow dynamical system (29) is rational in the marginal decision sense. □

**Remark 7.** As shown in the proof, the link-based DTD model (3) is also rational in the sense of the marginal decision rule.

**5. Numerical example**

This section illustrates the characteristics of the proposed MU-DTD model (29) by numerically approximating its trajectory. The numerical approximation of the continuous-time trajectory can be regarded as a discrete-time representation of DTD traffic flow evolution. Based upon the continuous-time MU-DTD model (29), the implementation of discrete-time MU-DTD model is as follows. Denote $x(t)$ as the link flow pattern at time step $t$. The link flow pattern at time step $t+1$, $x(t+1) = x(t) + \tau [z(t) - x(t)]$, where $z(t)$ solves optimization problem (26).

This approximation can be seen as a finite difference method with step size $\tau$. The DTD traffic flow evolution trajectory corresponding to the link-based model can be approximated by $x(t+1) = x(t) + \tau [y(t) - x(t)]$ where $y(t)$ solves optimization problem (4).

The proposed MU-DTD model (29) is applied to a three-by-three grid network with nine nodes and twelve links, whose topology is illustrated in Fig. 1. The network contains one origin-destination pair (node 1 to node 9), and the total demand is 2000. The number next to a link is its link number. In the numerical example, the link travel time function is of the BPR type:
\[ c_a(x) = c_a^0 \left[ 1 + b_a \left( \frac{x}{Q_a} \right)^4 \right] \]  

(50)

where \( Q_a \) denotes the nominal link capacity and parameter \( b_a \) indicates the weight of the volume/capacity ratio. In this example, all links have the same free flow travel time \( c_a^0 = 15 \) and nominal capacity \( Q_a = 1000 \). In the BPR link cost function, parameter \( b_a \) is set to be 0.3 for link 9, and \( b_a = 0.15 \) for all other links. As the BPR function is separable, the potential function \( U(t) = \sum_a \int_0^{x_a(t)} c(w) dw \) is used to represent the aggregate travel cost.

In all test scenarios, the initial link flow pattern is at user equilibrium. To illustrate the traffic flow evolution, assume that a 50% capacity reduction on link 1 takes place at time step 5. Notice that, for each time step \( t \), the “target flow” \( y(t) \) is required to construct the marginal disutility \( d(t) \) in the MU-DTD model (29).

The first test scenario compares the trajectories characterized by the link-based DTD model (3) and the MU-DTD model (29). The parameter \( \tau \) is set to 0.1 in both models. Parameter \( \beta \) in the optimization problem (4) is set to be 0.9 to determine the “target flow” \( y(t) \) in the link-based DTD model. For comparison, parameter \( \eta = \beta = 0.9 \) in the optimization problem (26), representing the case that drivers weigh the perceived marginal disutility in the same way as the perceived average cost in the link-based DTD model (3). In the first test scenario, parameter \( \mu \) is set to value 5, indicating the weight that drivers put on the marginal cost in the marginal
disutility $d(t)$. Fig. 2 illustrates the trajectories of the potential function $U(t)$, as well as the trajectories of link flow evolution on links 1 and 5.

![Graph](image)

**(a) Trajectories of the potential function $U(t)$**

![Graph](image)

**(b) Trajectories of flow evolution on link 1**

![Graph](image)

**(c) Trajectories of flow evolution on link 5**

**Fig. 2. Comparisons between the link-based DTD model and the MU-DTD model**

As shown in Fig. 2(a), where the x-axis value represents the time step and y-axis represents the value of the potential function $U(t)$, both trajectories characterized by the link-based and the MU-DTD models gradually converge to the new user equilibrium, as shown in Lemma 3 and Theorem 4. Comparing the link flow trajectories in Fig. 2(b) and 2(c), we can note that the link flow pattern provided by the MU-DTD model is closer to the new user equilibrium after the capacity reduction. It illustrates that the introduction of the marginal cost $\mu [c(x(t)) - c(y(t))]$ in the marginal utility DTD model can help stabilize the DTD link flow pattern towards user equilibrium. In particular, flows on links 1 and 5 reduce quickly as the marginal cost is considered in the MU-DTD model.
The second test scenario investigates the impacts of parameter $\mu$ on the MU-DTD model. A larger value of $\mu$ is used in the marginal disutility function (32). The larger value of $\mu$ increases the weight of the perceived marginal cost $\tilde{c}(x(t))$ in the marginal utility $d(t)$. Other parameters in the MU-DTD model are unchanged.

Fig. 3 shows the trajectory of the potential function $U(t)$ when $\mu = 25$. The result illustrates that as the value of $\mu$ increases, especially when it is greater than a certain threshold value $\alpha_{\text{max}}$, the traffic flow pattern diverges from the user equilibrium, as discussed in Remark 3. In this test scenario, the value of the potential function $U(t)$ increases significantly after the link capacity reduction. It indicates that drivers may overestimate the impact from other drivers’ route switching behavior. A detailed comparison of link flow evolution can help in better understanding the reason for the increase in the value of the potential function $U(t)$.

![Fig. 3. The evolution of aggregate travel cost under $\mu = 25$](image)

Fig. 4 compares the link flow evolution trajectories generated by the MU-DTD model under different values of $\mu$. When drivers can reasonably evaluate the potential impact of other drivers’ route switching behavior, e.g. $\mu = 5 < \alpha_{\text{max}}$, link flows can converge to UE. However, if drivers overestimate the potential impact of other drivers’ route switching behavior, flows could be attracted to links with high costs. As all drivers are assumed to be 1-step players in the proposed model, each driver believes that other drivers would switch to another link with lower cost. Hence, drivers tend to switch to links with high costs, as shown in Fig. 4 when $\mu = 25 > \alpha_{\text{max}}$. 

The two test scenarios highlight the need to further improve the proposed MU-DTD model. First, they show that traffic can evolve to UE only when drivers reasonably consider the impact of other drivers’ route switching behavior. If drivers overestimate the impact of others’ route switching behavior, the traffic flow pattern may significantly deviate from UE. Such effect can be seen as the result of overreaction to traffic information as discussed in Ben-Akiva et al. (1991). Second, they illustrate the shortcomings of the assumption of driver homogeneity. In the proposed MU-DTD model, all drivers are assumed to use 1-step strategic thinking. It can cause drivers to switch to links with high travel costs. However, cognitive experiments in behavioral game theory show that 1- and 2-step of strategic thinking are common and players can learn and update their thinking strategy (Costa-Gomes et al., 2001; Camerer et al., 2002; Camerer et al., 2004). This suggests the need to consider driver heterogeneity and learning process in the context of strategic thinking when modeling DTD traffic evolution.

6. Conclusions

In the DTD traffic flow evolution context, this paper proposes a conceptual shift in the definition of rationality which is inspired by a fundamental principle in economics: rational people make their decisions based on the difference between the marginal benefit and the marginal cost. Thereby, a new marginal utility DTD traffic evolution model is developed which can better address the behavioral tendency of the driver perceiving the cost sensitivity to link/route flow. In addition to the marginal benefit captured in classical DTD models, the marginal cost due to the route switching behavior is formulated in the proposed model.

The proposed MU-DTD model is built upon a dynamical system with an explicit representation of marginal utility. The perceived marginal cost function is specified using the
intermediate traffic flow pattern determined by the link-based DTD model proposed in He et al. (2010). The proposed model with the explicit representation of the perceived marginal cost function can maintain good mathematical properties, including the invariance property, asymptotic stability, and the equivalence between the stationary point and the DUE.

This study offers many future research directions. First, the marginal decision rule can be applied to other deterministic DTD traffic evolution models, either route-based models such as the one proposed in Kumar and Peeta (2015) or link-based models such as the flow splitting rate model (Smith and Mounce, 2011). Second, the proposed MU-DTD model may be extended to accommodate other driver perceptions of the marginal cost, such as the costs for updating the perception of network conditions, including obtaining, analyzing, and predicting traffic conditions. Third, the proposed MU-DTD model is derived based on the assumption that all drivers are one-step players. This assumption can be relaxed by assuming that only a few sophisticated and well-informed drivers are able to predict other drivers’ route choice. Fourth, the learning process can include the consideration of drivers’ update of their route switching strategies when they are heterogeneous k-step players. Fifth, the notion of marginal utility can be further applied to stochastic DTD traffic flow evolution processes. Finally, a calibrated and validated MU-DTD model can be used to develop effective traffic operational plans.

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**Lemma 6.** Let \( \mathbf{x}^* \) be a fixed point of the MU-DTD model (29) and \( \mathbf{x}(t) \) be a trajectory determined by the model. Suppose Assumptions 1 and 2 hold and parameter \( \mu \) in the marginal disutility function satisfies \( \mu \leq \alpha_{\text{max}} = (1-\nu)/(\bar{\rho}+\varepsilon) \) for given \( 0<\nu<1 \) and \( \varepsilon>0 \). Then, inequality:

\[
(\mathbf{x}(t)-\mathbf{x}^*)^T [\mathbf{x}(t)-\mathbf{z}(t)] \geq \left[ \nu - \frac{\gamma^2 (1+\mu^2 L^2)}{4} \right] \| \mathbf{x}(t)-\mathbf{y}(t) \|^2
\]

(A.1)

holds for all time \( t \).

**Proof.** Denote \( \mathbf{z}(t) = \text{Pr}_\Omega [ \mathbf{x}(t)-\gamma \mathbf{d}(\mathbf{x}(t)) ] \), and \( \mathbf{r}(t) = \mathbf{x}(t)-\mathbf{z}(t) \). From Facchinei and Pang (2003), Theorem 1.5.5 part (b):

\[
[ \mathbf{v} - \text{Pr}_\Omega (\mathbf{v}) ]^T [ \text{Pr}_\Omega (\mathbf{v}) - \mathbf{u} ] \geq 0, \quad \forall \mathbf{v} \in \mathbb{R}^n, \quad \forall \mathbf{u} \in \Omega.
\]

Let \( \mathbf{v} = \mathbf{x}(t)-\gamma \mathbf{d}(\mathbf{x}(t)) \) and \( \mathbf{u} = \mathbf{x}^* \) in the above inequality. Then:

\[
[ \mathbf{x}(t)-\gamma \mathbf{d}(\mathbf{x}(t)) - \text{Pr}_\Omega (\mathbf{x}(t)-\gamma \mathbf{d}(\mathbf{x}(t))) ]^T [ \text{Pr}_\Omega (\mathbf{x}(t)-\gamma \mathbf{d}(\mathbf{x}(t)))-\mathbf{x}^* ]
\]

\[
= [ \mathbf{r}(t)-\gamma \mathbf{d}(\mathbf{x}(t)) ]^T [ \mathbf{z}(t)-\mathbf{x}^* ] \geq 0.
\]

(A.2)
Due to Corollary 2 and $z(t) \in \Omega$:

$$\gamma \mathbf{d}(x^*)^T [z(t) - x^*] \geq 0. \quad (A.3)$$

Adding (A.2) to (A.3):

$$\mathbf{r}(t) - \gamma (\mathbf{d}(x(t)) - \mathbf{d}(x^*))^T [z(t) - x^*] \geq 0. \quad (A.4)$$

$$\begin{align*}
[r(t) - \gamma (d(x(t)) - d(x^*))]^T [z(t) - x^*] \\
= [r(t) - \gamma (d(x(t)) - d(x^*))]^T [z(t) - x(t) + x(t) - x^*] \\
= [r(t) - \gamma (d(x(t)) - d(x^*))]^T [-r(t) + (x(t) - x^*)] \\
= -\|r(t)\|^2 + (x(t) - x^*)^T r(t) + \gamma (d(x(t)) - d(x^*))^T r(t) - (d(x(t)) - d(x^*))^T (x(t) - x^*).
\end{align*}$$

As $x^*$ is an equilibrium point, $\mathbf{d}(x^*) = 0$. The above equation simplifies to:

$$-\|r(t)\|^2 + (x(t) - x^*)^T r(t) + \gamma (\mathbf{d}(x(t)) - \mathbf{d}(x^*))^T r(t) - \mathbf{d}(x(t))^T (x(t) - x^*) \quad (A.5)$$

Note that $y(t) = \Pr_{\Omega}(x(t) - c(x(t))) \in \Omega$, which satisfies $\mu c(x^*)^T [y(t) - x^*] \geq 0$, and:

$$\begin{align*}
[x(t) - \mu c(x(t)) - \Pr_{\Omega}(x(t) - \mu c(x(t)))]^T [\Pr_{\Omega}(x(t) - \mu c(x(t))) - x^*] \\
= [x(t) - \mu c(x(t)) - \Pr_{\Omega}(x(t) - \mu c(x(t)))]^T [\Pr_{\Omega}(x(t) - \mu c(x(t))) - x^*] \\
= 0.
\end{align*}$$

Adding these two inequalities leads to:

$$0 \leq [x(t) - y(t) - \mu c(x(t)) + \mu c(x^*)]^T [y(t) - x^*]$$

$$= [d(x(t)) + \mu (c(x^*) - c(y(t)))]^T [y(t) - x^*]$$

$$= d(x(t))^T [y(t) - x^*] + \mu (c(x^*) - c(y(t)))^T [y(t) - x^*].$$

Due to Assumption 1, $(c(y(t)) - c(x^*))^T [y(t) - x^*] > 0$. And:

$$\begin{align*}
d(x(t))^T [y(t) - x^*] &= d(x(t))^T [y(t) - x(t)] + d(x(t))^T [x(t) - x^*].
\end{align*}$$

Therefore,

$$\begin{align*}
d(x(t))^T [y(t) - x(t)] + d(x(t))^T [x(t) - x^*] \geq \mu (c(y(t)) - c(x^*))^T [y(t) - x^*] \geq 0.
\end{align*}$$

It implies that:
\[ d(x(t))^T [x(t)-x^*] \geq d(x(t))^T [x(t)-y(t)]. \]

The above inequality holds for all \( \mu > 0 \). Due to (A.5):

\[
-\|r(t)\|^2 + (x(t)-x^*)^T r(t) + \gamma d(x(t))^T r(t) - d(x(t))^T [x(t)-y(t)] \\
\leq -\|r(t)\|^2 + (x(t)-x^*)^T r(t) + \gamma d(x(t))^T r(t) - d(x(t))^T [x(t)-y(t)]
\]

Due to Lemma 5 and \( \mu \leq \alpha_{max} = (1-\nu)/(\bar{\rho}+\varepsilon) \),

\[
-d(x(t))^T [x(t)-y(t)] = -\left[ x(t)-y(t) - \mu (c(x(t))-c(y(t))) \right]^T [x(t)-y(t)] \\
\leq -\|x(t)-y(t)\|^2 + (1-\nu)\|x(t)-y(t)\|^2 \\
= -\nu\|x(t)-y(t)\|^2
\]

Therefore,

\[
-\|r(t)\|^2 + (x(t)-x^*)^T r(t) + \gamma d(x(t))^T r(t) - d(x(t))^T [x(t)-y(t)] \\
\leq (x(t)-x^*)^T r(t) - \|r(t)\|^2 + \gamma d(x(t))^T r(t) - \frac{\gamma^2}{4}\|d(x(t))\|^2 + \frac{\gamma^2}{4}\|d(x(t))\|^2 - \nu\|x(t)-y(t)\|^2
\]

Note that:

\[
\|d(x(t))\|^2 = \|x(t)-y(t)\|^2 - 2\mu (x(t)-y(t))^T (c(x(t))-c(y(t))) + \mu^2 \|c(x(t))-c(y(t))\|^2 \\
\leq (1+\mu^2L^2)\|x(t)-y(t)\|^2
\]

Thus,

\[
0 \leq -\|r(t)\|^2 + (x(t)-x^*)^T r(t) + \gamma d(x(t))^T r(t) - d(x(t))^T [y(t)-x(t)] \\
\leq (x(t)-x^*)^T r(t) + \frac{\gamma^2}{4}\left(1+\mu^2L^2\right)\|x(t)-y(t)\|^2 - \nu\|x(t)-y(t)\|^2
\]

Thus, \( (x(t)-x^*)^T r(t) \geq \left[ \nu - \frac{\gamma^2}{4}\left(1+\mu^2L^2\right) \right] \|x(t)-y(t)\|^2. \)