Entropy Weighted Average Method for the Determination of a Single Representative Path Flow Solution for the Static User Equilibrium Traffic Assignment Problem

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\textbf{A B S T R A C T}

The formulation of the static user equilibrium traffic assignment problem (UETAP) under some simplifying assumptions has a unique solution in terms of link flows but not in terms of path flows. Large variations are possible in the path flows obtained using different UETAP solution algorithms. Many transportation planning and management applications entail the need for path flows. This raises the issue of generating a meaningful path flow solution in practice. Past studies have sought to determine a single path flow solution using the maximum entropy concept. This study proposes an alternate approach to determine a single path flow solution that represents the entropy weighted average of the UETAP path flow solution space. It has the minimum expected Euclidean distance from all other path flow solution vectors of the UETAP. The mathematical model of the proposed entropy weighted average method is derived and its solution stability is proved. The model is easy to interpret and generalizes the proportionality condition of Bar-Gera and Boyce (1999). Results of numerical experiments using networks of different sizes suggest that the path flow solutions for the UETAP using the proposed method are about identical to those obtained using the maximum entropy approach. The entropy weighted average method requires low computational effort and is easier to implement, and can therefore serve as a potential alternative to the maximum entropy approach in practice.

Key words: user equilibrium, non-uniqueness, entropy weighted path flow solution

\section{1. Introduction}

\subsection{1.1 Background}

The user equilibrium traffic assignment problem (UETAP) can be formulated as an optimization problem using Beckmann’s transformation (Beckmann et al., 1956). A known limitation of the UETAP formulation is that under some simplifying assumptions (strictly increasing, additive and separable link cost functions) it has a unique solution in terms of link flows but not in terms of path flows (Sheffi, 1985). This is because multiple path flow combinations can lead to the same link flow vector. Bar-Gera and Luzon (2007) demonstrate a method to generate multiple user equilibrium (UE) path flow solutions by swapping flows between the paths of different origin-destination (O-D) pairs that use a common pair of diverge and merge points.

Several transportation planning and management applications require path flows or O-D specific link flows. Multiple path flow solutions are possible for the UETAP depending on the solution method used and the choice of the algorithmic parameters. Large variations are possible in the path flows obtained using different solution algorithms for the same input data, as shown by Larsson et al. (2004). This raises the question of the meaningfulness of a path flow solution for practical applications.

Past studies have sought to identify a single representative path flow (or O-D specific link flow) solution
for the UETAP using the concept of entropy maximization. It is based on the method of most probable values (MMPV) from statistical thermodynamics, for which two theoretical issues have been identified. First, the method relies on Stirling’s approximation for converting the entropy into a continuous function, and is subjected to the limitations of this approximation (Schrödinger, 1948, Chapter 1). Second, it assumes an overwhelmingly high probability of realization for the state with maximum entropy. However, if this assumption is violated, it can raise concerns related to the representativeness of the most probable state.

This study explores the method of mean values (MMV) from statistical thermodynamics to determine the entropy weighted average state as a single representative path flow solution for the UETAP. Insights from statistical thermodynamics suggest that while both MMV and MMPV may lead to the same solution, the MMV is theoretically more intuitive due to its representativeness (Schrödinger, 1948, Chapter 6). It leads to a simpler mathematical model that is easy to implement and can serve as an alternative to the entropy maximization method in practice.

The next section summarizes entropy maximization based approaches for finding a single UETAP path flow solution and the challenges associated with them.

1.2 Literature review

The first effort to uniquely define path flows for the UETAP is the maximum entropy user equilibrium (MEUE) formulation by Rossi et al. (1989). The assumption behind the MEUE formulation is that higher the entropy (measure of randomness) associated with a path flow solution vector, the higher the probability of its realization. Rossi et al. (1989) propose a two-step solution procedure. In the first step, precise UE link flows are obtained and the set of UE paths is enumerated. In the second step, the most likely path flows are obtained by solving the entropy maximization problem. They use an optimization software to illustrate numerical results but the analysis is limited to a small-scale network. The need for a good initial feasible solution and the scalability issue of the software used deter the use of their MEUE approach for large-scale networks. Lu and Nie (2010) showed that the MEUE solution is stable implying that that small changes in UETAP inputs lead to small changes in MEUE path flows.

Janson (1993) proposes an approximate method for finding the most likely O-D specific link flows. It determines an approximate MEUE solution as the linear combination of multiple O-D specific link flow solutions. Akamatsu (1997) points out that the solution method proposed by Janson (1993) is not consistent with the MEUE solution and proposes an entropy decomposition technique for finding the most likely O-D specific link flows based on Dial’s algorithm (Dial, 1971). However, his study focuses only on the theoretical formulation and not its implementation.

Some past studies have exploited dual methods for finding the solution to the MEUE problem. They include methods proposed by Bell and Iida (1997) and Larsson et al. (2004). Larsson et al. (2004) also establish that the MEUE is the limiting case of the stochastic user equilibrium (SUE) (Fisk, 1980) when travelers’ cost dispersion tends to zero. Further, they demonstrate the implementation of the dual method for small- to medium-sized real networks. However, the dual method has limitations that deter its deployment in practice, primarily in that the dual solution is typically primal infeasible during the iterative process. The feasibility of the primal problem is satisfied only when the dual solution reaches convergence. The numerical results of Larsson et al. (2004) suggest large computational times to obtain convergence for the dual method.

Bar-Gera (2006) proposes a primal method for finding the “most likely” path flows based on the behavioral interpretation of the proportionality condition (Bar-Gera and Boyce, 1999) for the MEUE problem. The proportionality condition states that travelers facing a choice set of two alternative segments
on their journey opt between these segments in the same proportion irrespective of their origin or destination. Bar-Gera (2010) proposes a solution algorithm labeled “TAPAS” that simultaneously solves the MEUE problem and the UETAP based on the proportionality condition. The key challenge for the methods based on the proportionality condition is the identification of feasible path flow adjustments without changing the total link flows and O-D flows (Bar-Gera, 2006, 2010). They rely upon the identification of paired alternative segments (PAS) and flow swap at the PAS for this purpose. Bar-Gera (2010) demonstrates the potential shortcomings of operating in the space of PAS and suggests remedies for them.

Mamun et al. (2011) show that the proportionality condition is exhibited by any SUE path flow solution, and hence it is natural to be observed in the MEUE solution as it represents its limiting case. Based on the findings of Leung and Yan (1997), Mamun et al. (2011) suggest that the probability of realization of the MEUE path flows may be negligibly small. They advocate using the mean of the UE path flow solutions instead of the MEUE path flows. They propose a sampling method to generate multiple path flow solution realizations for the UETAP, and the computation of their mean by assuming that all realizations have the same probability. There are two key limitations of this approach. First, assuming equal probability may not be reasonable though these probabilities are small. Second, the sample may not be representative of the entire solution space. Hence, the sampling method of Mamun et al. (2011) is unlikely to lead to the true mean of the solution space.

In summary, the following factors motivate this study. First, MMV is considered to be theoretically intuitive due to its representative nature (Schrödinger, 1948) but has not been previously explored to identify a single representative UETAP path flow solution. Second, it has the potential to provide a model which has a simpler form than the MEUE. Third, there are challenges related to the implementation of maximum entropy based approaches, such as: (i) the need for an initial feasible solution for the iterative solution procedure, (ii) the infeasibility of the dual solution at intermediate stages, and (iii) the challenge in finding effective PAS for the proportionality condition based methods.

1.3 Contribution and significance

This study seeks to determine the entropy weighted user equilibrium (EWUE) as the entropy weighted average of the path flow solution vectors in the entire UETAP solution space. It is based on the MMV proposed by Darwin and Fowler (1922). The EWUE has the minimum expected Euclidean distance from all other path flow vectors in the UETAP solution space, and hence can be viewed as being representative of the solution space.

As will be discussed in Section 3.2, the mathematical model of the EWUE generalizes the proportionality condition of Bar-Gera and Boyce (1999) for the PAS to the paths. That is, the proportionality condition for the PAS can be derived from the EWUE mathematical model. There are two key differences between the EWUE and the TAPAS algorithm (Bar-Gera, 2010). First, TAPAS operates in the space of PAS while EWUE operates directly in the space of paths. Second, TAPAS solves the UETAP and MEUE simultaneously. Hence, in TAPAS the proportionality condition is restored after every equilibration move towards UE (or flow swaps at PAS). By contrast, similar to the other MEUE models proposed in the past (Rossi et al., 1989; Bell and Iida, 1997; Bar-Gera, 2006), the EWUE is a post-processing step that is performed just once after the UETAP solution is determined in terms of link flows and UE paths.

In summary, there are four key contributions of this study. First, it provides an alternative approach to MEUE. Second, the EWUE model generalizes the proportionality condition of Bar-Gera and Boyce (1999) from the PAS to the paths. Third, the solution stability of EWUE is established. Fourth, the study derives calibration equations and provides implementation steps for calibrating the model parameters. The EWUE can be implemented using simple steps while incurring little computational time. The
implementation simplicity of the EWUE represents an important benefit over the MEUE model. Hence, it can act as a potential alternative to the MEUE method in practice.

The remainder of the paper is organized as follows. Section 2 provides the notation and an overview of the MEUE formulation. Section 3 presents the conceptual methodology and derivation of the proposed EWUE mathematical model. Section 4 provides the proof of stability of the EWUE solution. Section 5 discusses implementation aspects. Section 6 summarizes results of numerical experiments and their insights. Section 7 provides some concluding comments.

2. Preliminaries

This section provides the notation and defines the problem. It then presents an overview of the entropy maximization based solution methodology for the determination of the most likely path flow solution.

2.1 Notation

Let the transportation network of interest be represented by a strongly connected directed graph \( G(N, A) \), consisting of node set \( N \) and link set \( A \). Let \( W \) be the set of O-D pairs and \( K \) the set of used paths. The cardinality of sets \( A, W \) and \( K \) are \( l, m \) and \( n \), respectively. The set of used paths that connects the O-D pair \( w \in W \) is denoted as \( K^w \) and the travel demand for the O-D pair \( w \in W \) as \( d^w \). The flow on path \( k \) is denoted as \( f_k \) and the cost of traversing it as \( c_k \cdot f_k^w \) and \( c_k^w \), respectively, represent the flow and cost of path \( k \) connecting the O-D pair \( w \). The flow and the travel cost on link \( a \in A \) are denoted by \( x_a \) and \( g_a \), respectively. \( d \) is the vector of travel demand for all O-D pairs. \( f^w \) is the vector of path flows for O-D pair \( w \) and \( f \) is the vector of path flows for all O-D pairs. \( c^w \) is the vector of path costs for O-D pair \( w \) and \( c \) is the vector of path costs for all O-D pairs. \( x \) is the vector of link flows and \( g \) is the vector of link costs. The link-path incidence relationship is represented by the variable \( \delta_{ak} \) which takes the value 1 if link \( a \) is on path \( k \), and 0 otherwise. Similarly, the path-O-D incidence relationship is denoted as \( \phi_{kw} \). The link-path incidence relationship for all links and used paths is denoted by the matrix \( \Delta \), and the path-O-D incidence relationship for all paths and O-D pairs by \( \Phi \).

The Wardropian UE (Wardrop, 1952) assumes the familiarity of all travelers with the network and their homogenous rational behavior. In addition, let the following assumptions hold: (i) the O-D demand is constant and non-negative for all O-D pairs, (ii) \( g_a \) is a strictly increasing non-negative continuous function of \( x_a \), and (iii) there are no link interactions. Then, the UETAP can be formulated as the following optimization problem (Dafermos and Sparrow, 1969):

\[
\min \quad Z(x) = \sum_{a \in A} \int_{0}^{x_a} g_a(\omega) \, d\omega, \\
\text{s.t.} \quad x_a = \sum_k \delta_{ak} f_k, \forall a \in A, \\
\quad \quad \quad \quad d^w = \sum_k \phi_{kw} f_k, \forall w \in W,
\]

(1)

(2)

(3)
\[ f_k \geq 0, \forall k \in K. \] (4)

The UETAP represented by (1)-(4) has a unique solution in terms of link flows. However, it does not guarantee a unique solution in terms of path flows.

### 2.2 Problem definition

The uniqueness of link flows of the UETAP based on the aforementioned assumptions implies that the equilibrium link costs are also unique. Hence, as pointed out by Bar-Gera (2006), the set of UE paths are unique and their number is finite. Based on this, under some additional conditions, the most likely path flows for the UETAP can be obtained in two steps as proposed by Rossi et al. (1989). In the first step, the precise UE link flows and the set of UE paths can be obtained by solving the UETAP. Then, as the second step, the determination of the most likely path flows can be formulated as the following problem:

Determine the vector \( f \) that satisfies equations (5) and (6), given \( x, \Delta, d \) and \( \Phi \).

\[
\Delta f = x \tag{5}
\]

\[
\Phi^T f = d \tag{6}
\]

Equations (5) and (6) can be written in the combined matrix form as:

\[
\begin{bmatrix} \Delta \\ \Phi^T \end{bmatrix} f = \begin{bmatrix} x \\ d \end{bmatrix} \tag{7}
\]

Let the combined matrix \([\Delta; \Phi^T]\) formed by \( \Delta \) and \( \Phi \) in equation (7) be denoted as \( B \). Then, the system of equations represented by equation (7) has a unique solution if and only if the number of rows of matrix \( B \) is greater than or equal to its number of columns (which means \( l + m \geq n \) ) and matrix \( B \) has full rank. Typically, these two conditions are not satisfied for real networks and multiple solutions are possible for equation (7). Hence, some additional conditions are required for uniquely determining the solution vector \( f \). As discussed earlier, Rossi et al. (1989) propose the maximum entropy condition for identifying a single representative solution, and later many variants were developed. Next, a review of the maximum entropy approach is provided.

### 2.3 Review of the MEUE formulation

The MEUE-based methods in the literature are mostly based on the MEUE formulation by Rossi et al. (1989). This section presents a brief overview of this formulation. First, some necessary definitions are summarized.

**Definition 1.** A macro state (or flow condition) is defined as the allocation of path flows for all O-D pairs \( w \in W \) given the UE link flow vector \( x \) and travel demand vector \( d \). Alternatively, a macro state is defined by the path flow vector \( f \) (or more precisely by allocating values to each element of vector \( f \) ) that satisfies equations (5) and (6). Many macro states are possible for any given \( x \) and \( d \).

**Definition 2.** A micro state (or flow state) refers to the assignment of each network user (or traveler) to the paths for a given macro state or flow condition. Alternatively, a micro state is defined by assigning paths to each individual traveler/network user that satisfy the given path flow vector \( f \). Many micro states are possible for a macro state.
**Definition 3.** The *entropy* of any macro state is the measure of randomness or uncertainty of locating an individual random element (here, a network user). The higher the number of possible micro states for a macro state, the higher is the randomness or uncertainty of locating an individual network user in that macro state. Therefore, the entropy of a macro state can be defined as the number of possible micro states corresponding to that macro state.

Based on these definitions, the expression for entropy can be derived as follows. Suppose, there are $n^w$ UE paths between an O-D pair $w$. Then, using the basic principle of counting, the total number of network users $d^w$ can be assigned to these paths in the following number of ways:

$$S^w = \frac{d^w!}{f_1!f_2!...f_w!} = \frac{d^w!}{\prod_{k \in K^w} f_k !}. \quad (8)$$

The number of ways ($S^w$) of assigning individual network users to the UE paths for an O-D pair $w$ is independent of that of all other O-D pairs. Then, the total number of micro states (or entropy) of a macro state (defined by the path flow vector $f$) is:

$$S = \prod_{w \in W} \frac{d^w!}{\prod_{k \in K^w} f_k !}, \quad (9)$$

where $f_k$ is an element of the macro state defined by the vector $f$. Let $H$ be the set of all feasible path flow vectors defined by the sets of constraints represented by equations (5) and (6), i.e. $H$ is defined as: $H := \{ f | \Delta f = x; \Phi^T f = d \}$. The MEUE formulation of Rossi et al. (1989) can be represented by following maximization problem:

$$\max_{f \in H} S = \prod_{w \in W} \frac{d^w!}{\prod_{k \in K^w} f_k !}. \quad (10)$$

Taking the logarithm, the above maximization problem can be written as:

$$\max_{f \in H} \ln S = \sum_{w \in W} \left[ \ln d^w! - \sum_{k \in K^w} \ln f_k ! \right].$$

Dropping the constant term $d^w$ and changing the sign of second term, the optimization problem can be written as:

$$\min_{f \in H} - \ln S = \sum_{w \in W} \sum_{k \in K^w} \ln f_k !. \quad (11)$$

The optimization problem represented by equation (11) is difficult to solve for real-world networks in the current form. Therefore, Stirling’s approximation ($\ln x! \approx x \ln x - x$) is used to transform the equation (11) to the following suitable form:
\[
\min_{f \in H} -\ln S = \sum_{w \in W} \sum_{k \in K^w} f_k \ln f_k - d^w = \sum_{w \in W} \left( \sum_{k \in K^w} f_k \ln f_k - d^w \right).
\] (12)

Dropping the constant term \(d^w\) from equation (12), the MEUE formulation can be represented by the following minimization problem:

\[
\min_{f \in H} -\ln S = \sum_{w \in W} \sum_{k \in K^w} f_k \ln f_k .
\] (13)

The basic premise of the MEUE formulation comes from statistical thermodynamics which is based on two key assumptions. First, all micro states are equally likely. Therefore, higher the number of micro states (or entropy) of a macro state, higher is the probability of realization of that macro state. Second, entropy follows a sharp peak distribution over the possible macro states. From the real-world transportation network viewpoint, the first assumption implies that higher the number of ways in which drivers can select their paths without compromising their travel cost in a flow condition, higher is the probability of realization of that flow condition.

The second assumption implies that the probability of realization associated with the macro state with the highest entropy is so high that if we assume this macro state to be always realized we disregard only a very small fraction of realization probabilities over all other possible states (Schrödinger, 1948, Chapter 2). To evaluate the validity of this assumption for transportation networks, we consider two simple networks labeled as network 1 and 2. Network 1 is from Rossi et al. (1989) and network 2 is obtained by adding one additional origin node to network 1. The two networks along with the O-D demand are shown in Figs. 1 and 2. In these figures, the number above the link represents the link number. The UE link flows for links 1 and 2 are shown below these links. For other links, the UE flows are equal to the corresponding O-D demand. In both networks, two paths connect each O-D pair. One of these paths uses link 1 and the other uses link 2, as shown in Tables 1 and 2 for networks 1 and 2, respectively. The origins connected to destination D by these paths are shown in parentheses adjacent to the path numbers in Tables 1 and 2. Tables 1 and 2 also illustrate all possible flow conditions using integral flow swaps along with their entropies and probabilities of realizations for the respective networks. The entropy \(S_i\) for these macro states is computed using equation (9). Table 1 indicates that for network 1, the flow condition 2 has the highest entropy and is the most likely state with the probability of realization 0.6. Now, if we regard this state to be always realized we disregard 40 percent of the realization probabilities. In this case, there is dominance of this macro state over other states although neglecting other states may not be reasonable. But as shown in Table 2, for network 2 there are three flow conditions (shown bold in the table) that have similar probabilities of realizations. In this case, there is no dominance of any macro state. Therefore, the assumption of sharp peak distribution of entropy is problematic for this case as more than 71 percent of the states are ignored. Tables 1 and 2 also indicate that the probability of realization of the maximum entropy macro state is scale-dependent and falls sharply as the number of macro states increases. Hence, these examples illustrate the theoretical issue identified in statistical thermodynamics related to the representativeness of the most probable state, as discussed in Section 1.

![Fig. 1. Network 1.](image-url)
3. The EWUE model

This section presents the proposed conceptual methodology and the development of the EWUE mathematical model for obtaining a single representative path flow solution for the UETAP based on the insights from statistical thermodynamics.

3.1 Conceptual methodology

This study proposes that from a theoretical perspective, a path flow solution vector that has the minimum Euclidean distance from all other path flow solution vectors in the UETAP solution space \( H \), is a better representation of the solution space than the maximum entropy solution vector. Then, the minimum Euclidean distance can be used to identify a single representative path flow solution. Based on this criterion, the desired path flow solution vector \( \mathbf{f}^* \) can be determined by solving the following optimization problem:
\[ f^* := \arg \min_{f \in H} \sum_i \sqrt{\sum_k (f_{ki} - f_k^*)^2} p_i, \]  

(14)

where \( f_{ki} \) is the \( k \)th element of flow condition \( i \), \( f_k \) is the \( k \)th element of the path flow solution vector \( f \) that needs to be identified and \( p_i \) is the probability of realization of flow condition \( i \). The optimization problem represented by equation (14) can be expressed in the following equivalent form:

\[ \min_{f \in H} \sum_i \sum_k (f_{ki} - f_k^*)^2 p_i. \]  

(15)

Here, it is imperative to mention that if the entropy (and hence the \( p_i \)) follows a sharp peak distribution over the possible flow conditions (macro states), or if the entropy among the macro states is symmetrically distributed about its peak value, then the solution of the above optimization problem will practically coincide with the MEUE solution based on the optimization problem represented by equation (10).

Each term of objective function (15) corresponding to the path \( k \) for a given flow condition \( i \) is non-negative. Hence, its minimum can be obtained by minimizing the individual terms. Therefore, the optimization problem can be re-written as:

\[ \min_{f \in H} \sum_i (f_{ki} - f_k^*)^2 p_i, \forall k \in K. \]  

(16)

Using the first order condition for the minimum, we obtain:

\[ \sum_i \frac{d}{df_k} (f_{ki} - f_k^*)^2 p_i = 0, \forall k \in K \]  

(17)

\[ \Rightarrow \sum_i (f_{ki} - f_k^*) p_i = 0, \forall k \in K \]

\[ \Rightarrow f_k^* \sum_i p_i = \sum_i f_{ki} p_i, \forall k \in K. \]

The sum of the probabilities of realizations of the path flow vector \( f \) over all possible states is equal to 1. Therefore, elements of vector \( f^* \) can be determined as:

\[ f_k^* = \sum_i f_{ki} p_i, \forall k \in K. \]  

(18)

For \( f^* \) to be the solution of the optimization problem represented by the equations (15), it only remains to be established that the \( f^* \) obtained using equation (18) lies in the feasible space \( H \) i.e. it satisfies the set of constraints (5) and (6). The consistency of \( f^* \) with the set of constraints can be expressed in the form of the following property:

**Property 1.** (The consistency of expected path flows). If \( H \) is the solution space consisting of the solution vectors of the system of linear equations represented by equation (7), then the expectation of these solution vectors computed using equation (18) will also satisfy equation (7).
**Proof.** Property 1 is intuitive and can be established as follows. As stated earlier, the solution space $H$ is bounded by a set of linear equations; hence, it is convex. Since $0 < p_i \leq 1$ and $\sum_{i} p_i = 1$, the vector $f^*$ given by equation (18) represents the convex combination of the solutions of the system of linear equations represented by equation (7). Therefore, $f^*$ will also satisfy equation (7).

Now, recalling that the sets of constraints (5) and (6) is represented in a matrix form by equation (7), the existence of Property 1 confirms that the $f^*$ obtained by equation (18) is the solution to the optimization problem represented by equations (15).

Equation (18) implies that the determination of a single representative path flow vector for the UETAP based on the criteria of minimum expected Euclidean distance reduces to the determination of the expectation of the path flow vector over the entire feasible space bounded by the sets of equations (5) and (6), given the UE link flows, travel demand and the set of UE paths. This is analogous to the problem of finding the mean state using the method of mean values in statistical thermodynamics.

Using equation (18) and the basic premise from statistical thermodynamics that $p_i$ is proportional to $S_i$, the expectation of the flow of path $k$ can be expressed as:

$$E(f_k) = \frac{\sum_{i} f_{ki} S_i(f)}{\sum_{i} S_i(f)}, \forall k \in K, \tag{19}$$

where $S_i(.)$ is the entropy of the macro state $i$ and is defined by equation (9). Equation (19) represents the entropy weighted average (EWA) of the UE path flow solution space and is the conceptual expression for computing the representative path flow solution from the set of multiple path flow solutions for the UETAP. The corresponding path flow solution vector is labeled the EWUE solution vector.

### 3.2 Model development

This study uses the Darwin–Fowler method (Darwin and Fowler, 1922) to evaluate the numerator and denominator of equation (19) explicitly (see Ter Haar (1995) for a description of this method). Let the denominator and numerator of equation (19) be represented by $Q$ and $R$, respectively:

$$\sum_{i} S_i(f) = Q . \tag{20}$$

$$\sum_{i} f_{ki} S_i(f) = R . \tag{21}$$

The summations in equations (20) and (21) are carried over the entire feasible solution space bounded by the set of equations (5) and (6). Here, it is important to mention that the number of constraints represented by the set of equations (5) and (6) are typically much larger than in statistical thermodynamics where it is always two constraints. Therefore, the Darwin–Fowler method is customized in this study to evaluate the two summations. Hereafter, the subscript $i$ from $f_{ki}$ is dropped and $f_k$ is used to represent the $k^{th}$ element of flow condition $i$ for notational simplicity. Equations (5) and (6) can be represented in scalar form by the sets of equations (22) and (23) respectively as below:
\[ \sum_k \delta_{ak} f_k = x_a, \, \forall a \in A, \quad (22) \]
\[ \sum_k \phi_{kw} f_k = d^w, \, \forall w \in W. \quad (23) \]

The Darwin–Fowler method uses the concept of generating function introduced by Laplace (1820) for evaluating these summations. We define a generating function \( F(.) \) suitable for the above problem as:
\[ F(y_a, z_w, \delta_{ak}, \phi_{kw}) = \sum_{\text{all}(f)} S(f) \prod_{a \in A} y_a^{-\delta_{ak}} \prod_{w \in W} z_w^{\phi_{kw}}, \quad (24) \]

where the summation extends over all possible values of vector \( f \) without any constraint. Then, \( Q \) is given by the sum of the terms of \( F(.) \) whose \( y_a \) exponent equals to \( x_a \) and \( z_w \) exponent equals to \( d^w \). The sum of these terms can be obtained by using the Cauchy’s theorem of complex integration (for details of this method see Churchill and Brown (1960)) applied to each variable \( y_a \) and \( z_w \) as below:
\[ Q = \left( \frac{1}{2\pi i} \right)^{i+m} \oint dy_1 \oint dy_2 \ldots \oint dz_1 \oint dz_2 \ldots \prod_{a \in A} y_a^{-x_a} \prod_{w \in W} z_w^{-d^w} F(y_a, z_w, \delta_{ak}, \phi_{kw}). \quad (25) \]

In equation (25), \( y_a, \forall a \in A \) and \( z_w, \forall w \in W \) are assumed to be complex variables and integration is carried along a closed contour around the origin in counterclockwise direction. Similarly, \( R \) can be obtained as:
\[ R = \left( \frac{1}{2\pi i} \right)^{i+m} \oint dy_1 \oint dy_2 \ldots \oint dz_1 \oint dz_2 \ldots \prod_{a \in A} y_a^{-x_a} \prod_{w \in W} z_w^{-d^w} \left( \frac{1}{\ln y_a} \right) \frac{\partial}{\partial \delta_{ak}} F(y_a, z_w, \delta_{ak}, \phi_{kw}). \quad (26) \]

Equations (25) and (26) are formulated without making use of any approximations. These two equations along with equations (19)-(21) define the expectation of path flows over the feasible solution space bounded by the sets of equations (22) and (23). However, the integrals of equations (25) and (26) cannot be evaluated easily to obtain the exact solution. Therefore, an approximation technique, the method of steepest descent (Cauchy, 1847; Debye, 1909), is used to evaluate these integrals approximately. To use this method, the integrands of equations (25) and (26) need to be modified into a suitable form. We start with the simplification of entropy for this purpose. The entropy for flow condition \( i \) as per equation (9) is:
\[ S_i = \prod_{w \in W} \prod_{k \in K^w} d^w! \prod_{k \in K^w} f_k !. \]

Here, the numerator \( d^w! \) is a constant. Dropping this term, the entropy can be redefined as:
\[ S_i = \prod_{w \in W} \prod_{k \in K^w} \frac{1}{f_k !} = \prod_{k \in K} \frac{1}{f_k !}. \quad (27) \]

By defining a function \( \gamma(.) \) as \( \gamma(f_k) = 1 / (f_k !) \), we obtain:
\[ S_i = \prod_{k \in K} \gamma(f_k). \quad (28) \]
Using equation (28) for the $S_i$, the generating function $F$ of equation (24) can be written as:

$$F(\gamma_a, z_w, \delta_{ak}, \phi_{kw}) = \prod_{a \in A} \prod_{w \in W} \psi(y_a^\delta_a z_w^\phi_w),$$  \hspace{1cm} (29)

where function $\psi$ is defined as:

$$\psi(\eta) = \sum_{n=0}^{\infty} \gamma(n) \eta^n = e^\eta.$$  \hspace{1cm} (30)

To transform the integrands of equations (25) and (26) into suitable form, we define functions $u_a(.)$ and $u_w(.)$ such that they satisfy equations (31) and (32):

$$F(\gamma_a, z_w, \delta_{ak}, \phi_{kw}) y_a^{-x-1} = e^{-u_a(\gamma_a)}, \forall a \in A,$$  \hspace{1cm} (31)

$$F(\gamma_a, z_w, \delta_{ak}, \phi_{kw}) z_w^{-w-1} = e^{-u_w(\gamma_w)}, \forall w \in W.$$  \hspace{1cm} (32)

Focusing now only on integration over $y_a$, let:

$$I(y_a) = \frac{1}{2\pi i} \oint dy_a e^{-u_a(y_a)}.$$  \hspace{1cm} (33)

Using the method of steepest descent, the integral of equation (33) can be approximated as (Ter Haar, 1995):

$$I(y_a) \approx e^{-u_a(y_a)} \left[ \frac{1}{2\pi x_a u_a''(y_a)} \right]^{\frac{1}{2}},$$  \hspace{1cm} (34)

where $y_a^0$ satisfies the condition:

$$\frac{\partial u_a}{\partial y_a} = 0.$$  \hspace{1cm} (35)

Equation (35) can be written in terms of the generating function $F$ as:

$$y_a \frac{\partial F}{\partial y_a} - (x_a + 1) F = 0.$$  \hspace{1cm} (36)

Next, we follow the Darwin–Fowler method to approximate equation (36) by equation (37) and equation (34) by equation (38). The errors introduced by these approximations are insignificant when link flows and O-D demands are greater than 1. Further, these errors tend to cancel out (Ter Haar, 1995) as the terms approximated using equations (37) and (38) appear in both the numerator and the denominator of equation (19).

First, neglecting the 1 in the term within the parentheses in equation (36) we get:
\[ y_a \frac{1}{F} \frac{\partial F}{\partial y_a} \approx x_a. \] (37)

Second, the second factor in the expression of the integral \( I(y_a) \) in equation (34) is neglected. Therefore, the integral \( I(y_a) \) can be written as:

\[ I(y_a) \approx \exp_{a,u}(y_a^0) = F(y_a^0, z_w, \delta_{ak}, \phi_{kw})(y_a^0)^{-x_a^{-1}}. \] (38)

The integration over \( z_w \) can be performed in a similar manner and \( Q \) can be approximated as:

\[ Q \approx \prod_{a \in A}(y_a^0)^{-x_a^{-1}} \prod_{w \in W}(z_w^0)^{-d_w^{-1}} F(y_a^0, z_w^0, \delta_{ak}, \phi_{kw}), \] (39)

where \( y_a^0 \) is obtained by solving equation (37) and \( z_w^0 \) is obtained by solving equation (40):

\[ z_w \frac{1}{F} \frac{\partial F}{\partial z_w} = d_w. \] (40)

Similarly, \( R \) can be approximately written as:

\[ R \approx \prod_{a \in A}(y_a^0)^{-x_a^{-1}} \prod_{w \in W}(z_w^0)^{-d_w^{-1}} \left( \frac{1}{\ln y_a^0} \right) \frac{\partial}{\partial \delta_{ak}} F(y_a^0, z_w^0, \delta_{ak}, \phi_{kw}). \] (41)

Now, using equations (19)-(21), (39) and (41), the expectation of the flow of path \( k \) can be written as:

\[ E(f_k) = \prod_{a \in A} \prod_{w \in W} \left( \frac{1}{\ln y_a^0} \right) \frac{\partial}{\partial \delta_{ak}} F(y_a^0, z_w^0, \delta_{ak}, \phi_{kw}). \] (42)

Simplifying the notation, equation (42) can be written as:

\[ E(f_k) = \prod_{a \in A} \prod_{w \in W} \left( \frac{1}{\ln y_a^0} \right) F^0 \frac{\partial}{\partial \delta_{ak}} F^0, \] (43)

where \( F^0 = F(y_a^0, z_w^0, \delta_{ak}, \phi_{kw}) \). Using equations (29) and (30), the generating function \( F \) can be written as:

\[ F(y_a, z_w, \delta_{ak}, r_{kw}) = \prod_{a \in A} \prod_{w \in W} \exp(y_a^0, z_w^0). \] (44)

Now, partially differentiating both sides of equation (44) with respect to \( \delta_{ak} \), we obtain:

\[ \frac{\partial F}{\partial \delta_{ak}} = \left[ \prod_{a \in A} \prod_{w \in W} \exp(y_a^0, z_w^0) \right] \frac{\partial}{\partial \delta_{ak}} (y_a^0, z_w^0) \]
\[
\Rightarrow \frac{\partial F}{\partial \delta_{ak}} = F \cdot \varphi_{w_a} \cdot \frac{\partial}{\partial \delta_{ak}} (y_a^{\delta_{ak}}) = F \cdot \varphi_{w_a} (\ln y_a) y_a^{\delta_{ak}} \\
\Rightarrow \left( \frac{1}{\ln y_a} \right) \frac{1}{F} \frac{\partial F}{\partial \delta_{ak}} = \varphi_{w_a} y_a^{\delta_{ak}} .
\]

(45)

Using equations (43) and (45), we obtain:

\[
E(f_k) = \prod_{a \in A} (y_a^{\delta_{ak}}) \prod_{w \in W} (z_{w}^{\theta_w}) .
\]

(46)

Taking the logarithm of both sides of equation (44), we obtain:

\[
\ln F = \sum_{a \in A} \sum_{w \in W} y_a^{\delta_{ak}} z_{w}^{\theta_w} .
\]

(47)

Now, partially differentiating both sides of equation (47) with respect to \( y_a \), we obtain:

\[
\frac{\partial (\ln F)}{\partial y_a} = \sum_{w \in W} \varphi_{w_a} \frac{\partial}{\partial y_a} (y_a^{\delta_{ak}}) .
\]

(48)

The parameter \( \delta_{ak} \) can have one of the two values from the set \{0, 1\}. Consider the two cases based on this. First, assume that \( \delta_{ak} \) is equal to 0; then:

\[
\frac{\partial}{\partial y_a} (y_a^{\delta_{ak}}) = \frac{\partial}{\partial y_a} (y_a)^0 = 0 = \delta_{ak} .
\]

(49)

Next, assume that \( \delta_{ak} \) is equal to 1; then:

\[
\frac{\partial}{\partial y_a} (y_a^{\delta_{ak}}) = \frac{\partial}{\partial y_a} (y_a)^1 = 1 = \delta_{ak} .
\]

(50)

From equations (49) and (50), we obtain:

\[
\frac{\partial}{\partial y_a} (y_a^{\delta_{ak}}) = \delta_{ak} .
\]

(51)

Using equations (48) and (51), we obtain:

\[
\frac{1}{F} \frac{\partial F}{\partial y_a} = \delta_{ak} \sum_{w \in W} z_{w}^{\theta_w} .
\]

(52)

Substituting the value of \((1 / F)(\partial F / \partial y_a)\) from equation (52) and then solving equation (37), we obtain \( y_a^0 \) as:
\[
\gamma_a = \frac{x_a}{\delta_{ak} \sum_{w \in W} (z_w^0)^{\phi_{kw}}}. \tag{53}
\]

Similarly, \(z^0_w\) can be obtained by solving equation (40) as:
\[
z^0_w = \frac{d^w}{\phi_{kw} \sum_{a \in A} (y_a^0)^{\phi_{kw}}}. \tag{54}
\]

Substituting the values of \(y_a^0\) and \(z^0_w\) from equations (53) and (54) into equation (46), we obtain:
\[
E(f_k) = \prod_{a \in A} \left[ \frac{x_a}{\delta_{ak} \sum_{w \in W} (z^0_w)^{\phi_{kw}}} \right]^{\delta_{ak}} \prod_{w \in W} \left[ \frac{d^w}{\phi_{kw} \sum_{a \in A} (y_a^0)^{\phi_{kw}}} \right]^{\phi_{kw}}.
\tag{55}
\]

In equation (55), the factors of the products are unity when parameters \(\delta_{ak}\) and \(\phi_{kw}\) take the value zero. At the same time, the denominators of these factors do not change when these parameters assume the value 1. Hence, these two parameters can be omitted from the denominators. In addition, we denote in equation (55) parameters \(\alpha_a\) and \(\beta_w\) for notational simplification as:
\[
\alpha_a = \frac{1}{\sum_{w \in W} (z^0_w)^{\phi_{kw}}}, \tag{56}
\]
\[
\beta_w = \frac{1}{\sum_{a \in A} (y_a^0)^{\phi_{kw}}}. \tag{57}
\]

Then, equation (55) becomes:
\[
E(f_k) = \prod_{a \in A} [\alpha_a x_a]^{\delta_{ak}} \prod_{w \in W} [\beta_w d^w]^{\phi_{kw}}. \tag{58}
\]

Utilizing the fact that a path \(k\) can be an element of path set (say \(K\)) of only one O-D pair (say \(j\)), we have:
\[
\phi_{kw} = \begin{cases} 
1 & \text{when } w = j \\
0 & \forall w \in W \setminus \{j\}.
\end{cases} \tag{59}
\]

Hence, if \(f_k \in K\) then, by using equation (59), equation (58) reduces to the following form:
Equation (60) represents the functional form of the proposed EWUE model for finding the single representative path flow vector. According to the mathematical form of the model, the EWUE flow of a path is proportional to the travel demand of the O-D pair connected by that path and proportional to the flows of those links that lie on that path. In this sense, the proposed EWUE model generalizes the proportionality condition of Bar-Gera and Boyce (1999) for the PASs to the paths. In equation (60), $\alpha$ and $\beta$ are the balancing parameters that needs to be calibrated.

The proportionality condition for the PAS proposed by Bar-Gera and Boyce (1999) can be derived from the EWUE model as follows. Consider a PAS between nodes E and F shown as the dotted links in Fig. 3. Assume that AC and BD are two O-D pairs that are connected through paths using this PAS. In Fig. 3, the link numbers are shown above the links and the link flows are shown below the links. There are four paths (two paths for each O-D pair) that connect the two O-D pairs as defined by the following link sequences: (i) Path 1 connecting O-D pair AC as 1-3-5, (ii) Path 2 connecting O-D pair AC as 1-4-5, (iii) Path 3 connecting O-D pair BD as 2-3-6, and (iv) Path 4 connecting O-D pair BD as 2-4-6. Using the EWUE model represented by equation (60), the flows of these paths are:

\begin{align*}
 f_1^{AC} &= \beta_{AC} d^{AC} \alpha_1 x_1 \alpha_3 x_3 \alpha_5, \\
 f_2^{AC} &= \beta_{AC} d^{AC} \alpha_1 x_1 \alpha_4 x_4 \alpha_5, \\
 f_3^{BD} &= \beta_{BD} d^{BD} \alpha_2 x_2 \alpha_3 x_3 \alpha_6 x_6, \\
 f_4^{BD} &= \beta_{BD} d^{BD} \alpha_2 x_2 \alpha_4 x_4 \alpha_6 x_6.
\end{align*}

The ratios of the flows of the paths using the PAS between E and F for the two O-D pairs are:

\begin{align*}
 \frac{f_1^{AC}}{f_2^{AC}} &= \frac{\alpha_3 x_3}{\alpha_4 x_4}, \quad \text{and} \quad \frac{f_3^{BD}}{f_4^{BD}} = \frac{\alpha_3 x_3}{\alpha_4 x_4}.
\end{align*}

The equality of ratios of the flows of the paths using the PAS for the two O-D pairs leads to the proportionality condition proposed by Bar-Gera and Boyce (1999):

\begin{align*}
 \frac{f_1^{AC}}{f_2^{AC}} &= \frac{f_3^{BD}}{f_4^{BD}}.
\end{align*}

![Fig. 3. A paired alternative segment.](image)

The mathematical model represented by equation (60) provides the expectation of the path flow vector (defined by equation (19)) over the entire solution space bounded by equations (5) and (6). This leads to an interesting property of the model:

**Property 2.** (The generalized consistency of expected path flows) Given the link flow vector $x$, the O-D trip demand vector $d$ and the set of used paths defined by matrices $\Delta$ and $\Phi$, the expectation of the path...
flow vector over the solution space $H$ defined by equations (5) and (6) can be obtained by equation (60) irrespective of whether vector $x$ represents the Wardropian UE.

**Proof.** The model represented by equation (60) is derived without using the UE condition (the equivalency of the costs of used paths for an O-D pair). The development of this model only uses the travel demand, definition of the used paths, vector of link flows and the flow conservation constraints (represented by equations (5) and (6)) as the given conditions. Hence, the above formulation is valid irrespective of the link flow vector satisfying UE.

Property 2 implies that the proposed model can be used to estimate path flows corresponding to any link flow vector, O-D trip demand and the set of used paths. In this sense, a potential application of this model can be to estimate the expected path flows corresponding to the observed real-world link counts.

### 3.3 Calibration of the model parameters

The model parameters $\alpha$ and $\beta$ need to be calibrated for its implementation in practice. The expressions for the calibration of these parameters can be obtained by noting that the expected path flow vector must satisfy the sets of equations (22) and (23). Using equation (22), we obtain:

$$
\sum_k \delta_{ak} E(f_k) = x_a, \forall a \in A
$$

Using equations (60) and (61), we obtain:

$$
\sum_k \delta_{ak} \left( \beta_j d^1 \right) \prod_{a \in A} \left[ \alpha_a x_a \right]^{\delta_{ak}} = x_a, \forall a \in A,
$$

where $j$ is the O-D pair connected by path $k$. Let $K_a$ denote the set of paths that contain link $a$; then:

$$
\delta_{ak} = \begin{cases} 
1 & \forall k \in K_a \\
0 & \forall k \in K \setminus K_a 
\end{cases}
$$

Using equations (62) and (63), we obtain:

$$
\sum_{k \in K_a} \left( \beta_j d^1 \right) \prod_{\sigma \in A \setminus \{a\}} \left[ \alpha_{\sigma} x_{\sigma} \right]^{\delta_{ak}} = x_a, \forall a \in A.
$$

The sum in equation (64) is carried over only those paths $k$ that contain link $a$. Hence, all terms of this sum will have a factor $\alpha_a x_a$ for which the exponent $\delta_{ak}$ equals 1. Separating out this factor, we obtain:

$$
(\alpha_a x_a) \sum_{k \in K_a} \left( \beta_j d^1 \right) \prod_{\sigma \in A \setminus \{a\}} \left[ \alpha_{\sigma} x_{\sigma} \right]^{\delta_{ak}} = x_a, \forall a \in A.
$$

Canceling out the common factor $x_a$, the calibration equation of parameter $\alpha_a$ is obtained as:

$$
\alpha_a = \frac{1}{\sum_{k \in K_a} \left( \beta_j d^1 \right) \prod_{\sigma \in A \setminus \{a\}} \left[ \alpha_{\sigma} x_{\sigma} \right]^{\delta_{ak}}}, \forall a \in A
$$
where \( j : \{ j \in W \text{ and } k \in K^j \} \).

Similarly, using equation (23) we obtain:

\[
\sum_k \phi_{kw}E(f_k) = d^w, \forall w \in W
\]  

(67)

\[ \Rightarrow \sum_k \phi_{kw}E(f_k) + \sum_{k \in K^w} \phi_{kw}E(f_k) = d^w, \forall w \in W \]

\[ \Rightarrow \sum_{k \in K^w} E(f_k) + 0 = d^w, \forall w \in W. \]

Substituting the expression for \( E(f_k) \) from equation (60) in the above equation, we obtain:

\[
\sum_{k \in K^w} \left( \beta_w d^w \right) \prod_{a \in A} \left[ \alpha_a x_a \right]^\delta_a = d^w, \forall w \in W.
\]

(68)

In equation (68), \( w \) represents the O-D pair connected by path \( k \). After canceling out the common factor \( d_j \), the calibration equation of parameter \( \beta_w \) is obtained as:

\[
\beta_w = \frac{1}{\sum_{k \in K^w} \prod_{a \in A} \left[ \alpha_a x_a \right]^\delta_a}, \forall w \in W.
\]

(69)

Equations (66) and (69) collectively imply that parameters \( \alpha_a \) and \( \beta_w \) are interdependent and need to be calibrated using an iterative procedure. This study proposes the following procedure for parameter calibration and computation of the EWUE path flows based on equations (60), (66) and (69):

**Step 1:** Set \( n=0 \) and start by assuming \( \alpha_a = 1, \forall a \in A \) and \( \beta_w = 1, \forall w \in W \).

**Step 2:** Compute \( \alpha_a x_a, \forall a \in A \).

**Step 3:** Compute for each path \( k \), \( \tau_k = \prod_{a \in A} \left[ \alpha_a x_a \right]^\delta_a \).

**Step 4:** Compute for each path \( k \), \( \mu_k = \tau_k \beta_j d^j \), where \( j : \{ j \in W \text{ and } k \in K^j \} \), that is, \( j \) is the O-D pair connected by path \( k \).

**Step 5:** Compute \( \theta_a = \left( 1 / \alpha_a x_a \right) \times \sum_{k \in K_a} \mu_k, \forall a \in A \), where \( K_a \) is the set of paths that contain link \( a \).

**Step 6:** Set \( n=n+1 \) and compute the new value of the parameter \( \alpha_a \) as:

\[
\alpha_a \rightarrow (1 - ss\text{size})\alpha_a + ss\text{size}(1 / \theta_a), \forall a \in A,
\]

where \( ss\text{size} \) is the step size for updating the parameter \( \alpha \). This study uses a \( ss\text{size} \) value of 0.5.

**Step 7:** Compute \( \alpha_a x_a, \forall a \in A \).

**Step 8:** Compute for each path \( k \), \( \tau_k = \prod_{a \in A} \left[ \alpha_a x_a \right]^\delta_a \).

**Step 9:** Compute \( \nu_w = \sum_{k \in K^w} \tau_k, \forall w \in W \).
Step 10: Compute the new value of the parameter \( \beta_w \) as \( \beta_w = 1 / \nu_w, \forall w \in W \).

Step 11: Compute, for each path \( k \), the new path flows as:

\[
(f^w_k)_n = \tau_k \beta wd^w.
\]

Step 12: If \( n > 0 \), then compute error as the average absolute difference between the new and previous values of path flows:

\[
\text{error} = \frac{1}{\sum d^w} \left( \sum_{k \in K} |\mu_k - f_k| \right).
\]

Step 13: If the error is less than a pre-specified threshold limit \( \varepsilon \) (assumed in this study as 1.0E-07), save the path flows and the values of the parameters (\( \alpha \) and \( \beta \)) and stop. Otherwise update \( \mu \) for each path \( k \) as \( \mu_k = f_k \), and go to Step 5.

4. Stability of the EWUE solution

This section presents the stability analysis of the proposed EWUE solution. Solution stability here implies that small changes in the inputs to the UETAP should lead to small changes in the EWUE path flow solution. Solution stability is a desirable property in practice, especially for scenario analysis where alternative scenarios of transportation improvement projects that differ by little in terms of added links or their capacities need to be evaluated.

The inputs for the UETAP are travel demand \( (d) \) and the parameters of the link cost function \( g \). Let us consider a parameter \( \xi \) that takes values in the set \( \Xi \in \mathbb{R}^\sigma \), where \( \sigma \) is the number of parameters in the link cost function \( g \). Then, the link cost vector \( g(\xi, x) \) is a function of link flows \( x \) and parameters \( \xi \). As mentioned in Section 3, EWUE uses the travel demand \( d \) and the link flow solution of the UETAP \( (x^{UE}) \) as inputs. However, according to the UETAP formulation (equations (1)-(4)), \( x^{UE} \) is a function of \( (d, \xi) \). Hence, the EWUE path flow \( f^* \) can be assumed to be a function of \( (d, \xi) \). The stability of the EWUE solution then implies that small changes in the \( (d, \xi) \) lead to small changes in \( f^* \). Thereby, the EWUE path flow \( f^* \) will be stable if and only if \( f^* \) is bounded, unique and continuous with respect to \( (d, \xi) \).

The non-negativity constraint (4) and the flow conservation constraint (3) together imply that \( 0 \leq f_k^* \leq d^w \) and therefore, \( f^* \) is bounded. Lu and Nie (2010) have proved that under the assumption of strict monotonicity of the link cost functions, the UE link flows \( (x^{UE}) \) are a continuous function of \( (d, \xi) \) and the set of UE path flows \( (H) \) are a continuous multifunction of \( (d, \xi) \). The EWUE path flow \( f^* \) is one of the possible path flow solutions (represented by \( H \)) of the UETAP; that is, \( f^* \in H \). Therefore, \( f^* \) is also continuous with respect to \( (d, \xi) \). Now, the only condition that needs to be established for the stability of the EWUE path flow is the uniqueness of the EWUE path flow vector \( f^* \). The uniqueness property of the EWUE path flow vector \( f^* \) can be established as follows.

**Property 3.** (The uniqueness of EWUE path flows). Under the assumption of monotonically increasing and separable link cost functions, the EWUE path flow vector \( f^* \) is unique; that is, \( f^* \) is a single-valued function of the UETAP input parameters \( (d, \xi) \).
Proof. The input to the EWUE model are travel demand \((d)\), UE link flows \((x^{UE})\) and the UE path set \(K\). As stated in Section 2, under the assumption of strict monotonicity and separability of the link cost functions, the link flow solution of the UETAP is unique for a given \((d, \xi)\). Therefore, the UE link costs and the UE path set are also unique. The EWUE path flow \(f^*\) is the solution of the minimization problem represented by equation (15). The objective function of this minimization problem is convex. The feasible region \(H\) is bounded by the sets of linear constraints (5) and (6). Hence, the feasible region is also convex. Therefore, this optimization problem has a unique minimum at \(f^*\). Thereby, the EWUE path flow vector \(f^*\) is a single-valued function of the UETAP input parameters \((d, \xi)\). ■

This proves the stability of the EWUE solution.

5. Implementation aspects

A key need for the implementation of the proposed EWUE model is the requirement of the UE path set. While theoretically the UE path set is unique, it is difficult to obtain in practice. This is because an exact UE solution is difficult to achieve in practice for real networks. Hence, this study relies on generating a UE solution which is close to the exact solution. Larsson et al. (2004) present a method for generating the set of paths using the UE link costs and criteria for filtering the UE path set. In this study, the paths are generated along the iterative UETAP solution process, and criteria similar to Larsson et al. (2004) are used for filtering the UE path set. The slope-based path shift-propensity algorithm (2014), labeled as SPSA, is used for this purpose. The SPSA solves the UETAP problem iteratively. At each iteration, it updates the path sets of all O-D pairs simultaneously and then equilibrates one O-D pair at a time sequentially. The flow update mechanism of the SPSA divides the set of feasible paths for an O-D pair into costlier and cheaper sets by calculating a threshold limit based on the maximum cost difference between the path costs and using a proximity parameter. It uses the slopes of the cost function in the flow update process to determine the move direction, and shifts flows from the costlier paths to the cheaper paths for each O-D pair. Details of the mathematical derivation of the SPSA and its proof of convergence are provided in Kumar and Peeta (2014). The step size for the move direction is obtained by a line search technique. The SPSA is used for generating paths and computing the UE link flows. The shortest path for each O-D pair is generated at each iteration of the SPSA and included in the path set if it is not already present in it. The SPSA is initialized using the all-or-nothing assignment with free flow travel costs and then the path set and link flows are updated iteratively till convergence (relative gap of 1.0E-14) is achieved. The generated paths are filtered to obtain the approximate UE path set. The filtering process involves two criteria. First, the paths with non-zero flow are selected as the UE paths. The residual flows arising due to computation precision errors can be problematic for filtering the paths based on this criterion. To overcome this issue, paths whose flows are less than 1.0E-08 times the travel demands of their O-D pairs are excluded from the UE set. As the second criterion, the paths having costs within the band of 0.1% of the minimum travel cost of the respective O-D pairs are also included in the UE path set. Here, it is imperative to mention that while this study uses the SPSA to generate the UE path set and determine the UE link flows simultaneously, the EWUE implementation is indifferent to the method used to generate the UE paths (for example, from the UE link costs similar to Larsson et al. (2004)).

6. Numerical experiments

The numerical experiments are performed using four test networks. The first network (from Lu and Nie, 2010), labeled here as network 3, is a small-scale network and facilitates the direct comparison of the MEUE and EWUE path flow solution vectors. The other three networks are of larger scale and obtained from the website “Transportation Network Test Problems” (Bar-Gera). However, the O-D demand matrix
used for the Chicago Regional network is different from that of the website. Table 3 provides the characteristics of these networks. The link costs for these test networks are determined using BPR functions without considering toll and travel distance factors. The UE link flows and paths for these networks are obtained by implementing the SPSA. The SPSA, the EWUE model, and the calibration steps were coded in C++ and implemented on a high-performance computing cluster with 48 cores and 192 GB of memory per node. The MEUE path flows were obtained by solving the maximum entropy formulation (represented by equations (13), (5) and (6)) using the constrained optimization interior-point algorithm in MATLAB, implemented on the same computing platform. MEUE is assumed to converge if the change in the objective function (equation (13)) is less than or equal to 1.0E-3.

Table 3  
Characteristics of the test networks.

<table>
<thead>
<tr>
<th>Network</th>
<th>Nodes</th>
<th>Links</th>
<th>Zones</th>
<th>O-D pairs with non-zero demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>Network 3</td>
<td>13</td>
<td>16</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Anaheim</td>
<td>416</td>
<td>914</td>
<td>38</td>
<td>1,416</td>
</tr>
<tr>
<td>Chicago Sketch</td>
<td>933</td>
<td>2,950</td>
<td>387</td>
<td>93,135</td>
</tr>
<tr>
<td>Chicago Regional</td>
<td>12,982</td>
<td>39,018</td>
<td>1,790</td>
<td>3,168,206</td>
</tr>
</tbody>
</table>

The topology of network 3 is shown in Fig. 4. The numbers next to the links represent the corresponding link numbers. Nodes 1 and 2 are the two origins and the node 13 is the only destination in this network. The travel demands for the O-D pairs 1-13 and 2-13 are each equal to 5000. The link costs are computed using the BPR function:

\[ g_a(x_a) = g^0_a \left[ 1 + \lambda \left( \frac{x_a}{\lambda} \right)^\rho \right], \] (70)

where \( g^0_a \) and \( \lambda_a \) are, respectively, the free flow travel cost and the capacity of link a. \( \lambda \) and \( \rho \) are the parameters of the BPR function. The values of parameters for \( \lambda \) and \( \rho \) are assumed to be 0.15 and 4, respectively, for all of the test networks. The link properties of network 3 are shown in Table 4. The link properties of the other three test networks can be obtained from the website “Transportation Network Test Problems” (Bar-Gera).

We consider three scenarios in addition to the unperturbed scenario (the base case) for comparing the MEUE and EWUE flows for network 3. In Scenario 1, the capacity of link 5 is increased by 20% to 4800. It simulates the addition of a lane to this link. In Scenario 2, the travel demand for O-D pair 1-13 is
increased to 9990 and that of O-D pair 2-13 is decreased to 10. It simulates the shifting of residences from zone 2 to zone 1 due to the continued threat of flooding. Scenario 3 is the opposite to Scenario 2; here the travel demand for O-D pair 1-13 is decreased to 10 and that of O-D pair 2-13 is increased to 9990.

Table 4

<table>
<thead>
<tr>
<th>Link number</th>
<th>From-To</th>
<th>Capacity (veh/hour)</th>
<th>Free flow travel time (minutes)</th>
<th>UE link flow (veh/hour)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Base case</td>
</tr>
<tr>
<td>1</td>
<td>1-10</td>
<td>4.99E+09</td>
<td>0.00</td>
<td>5000.0</td>
</tr>
<tr>
<td>2</td>
<td>2-12</td>
<td>4.99E+09</td>
<td>0.00</td>
<td>5000.0</td>
</tr>
<tr>
<td>3</td>
<td>3-4</td>
<td>4000</td>
<td>0.25</td>
<td>4798.4</td>
</tr>
<tr>
<td>4</td>
<td>3-5</td>
<td>6000</td>
<td>1.00</td>
<td>201.6</td>
</tr>
<tr>
<td>5</td>
<td>4-5</td>
<td>4000</td>
<td>0.25</td>
<td>7327.7</td>
</tr>
<tr>
<td>6</td>
<td>4-6</td>
<td>6000</td>
<td>1.00</td>
<td>2470.7</td>
</tr>
<tr>
<td>7</td>
<td>5-6</td>
<td>4000</td>
<td>0.25</td>
<td>4863.7</td>
</tr>
<tr>
<td>8</td>
<td>5-7</td>
<td>6000</td>
<td>1.00</td>
<td>2665.6</td>
</tr>
<tr>
<td>9</td>
<td>6-7</td>
<td>4000</td>
<td>0.25</td>
<td>7334.4</td>
</tr>
<tr>
<td>10</td>
<td>7-8</td>
<td>4000</td>
<td>0.25</td>
<td>6483.6</td>
</tr>
<tr>
<td>11</td>
<td>7-9</td>
<td>6000</td>
<td>1.00</td>
<td>3516.4</td>
</tr>
<tr>
<td>12</td>
<td>8-9</td>
<td>4000</td>
<td>0.25</td>
<td>6483.6</td>
</tr>
<tr>
<td>13</td>
<td>9-11</td>
<td>8000</td>
<td>0.50</td>
<td>10000.0</td>
</tr>
<tr>
<td>14</td>
<td>10-4</td>
<td>8000</td>
<td>0.25</td>
<td>5000.0</td>
</tr>
<tr>
<td>15</td>
<td>11-13</td>
<td>4.99E+09</td>
<td>0.00</td>
<td>10000.0</td>
</tr>
<tr>
<td>16</td>
<td>12-3</td>
<td>8000</td>
<td>0.25</td>
<td>5000.0</td>
</tr>
</tbody>
</table>

Table 5

<table>
<thead>
<tr>
<th>Path number</th>
<th>O-D</th>
<th>EWUE flow</th>
<th>MEUE flow</th>
<th>Path as the link sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1-13</td>
<td>817.417</td>
<td>817.420</td>
<td>1-14-6-9-10-12-13-15</td>
</tr>
<tr>
<td>2</td>
<td>1-13</td>
<td>443.340</td>
<td>443.340</td>
<td>1-14-6-9-11-13-15</td>
</tr>
<tr>
<td>3</td>
<td>1-13</td>
<td>1566.050</td>
<td>1566.100</td>
<td>1-14-5-7-9-10-12-13-15</td>
</tr>
<tr>
<td>4</td>
<td>1-13</td>
<td>849.371</td>
<td>849.370</td>
<td>1-14-5-7-9-11-13-15</td>
</tr>
<tr>
<td>5</td>
<td>1-13</td>
<td>858.308</td>
<td>858.310</td>
<td>1-14-5-8-10-12-13-15</td>
</tr>
<tr>
<td>6</td>
<td>1-13</td>
<td>465.518</td>
<td>465.520</td>
<td>1-14-5-8-11-13-15</td>
</tr>
<tr>
<td>7</td>
<td>2-13</td>
<td>784.458</td>
<td>784.460</td>
<td>2-16-3-6-9-10-12-13-15</td>
</tr>
<tr>
<td>8</td>
<td>2-13</td>
<td>425.464</td>
<td>425.460</td>
<td>2-16-3-6-9-11-13-15</td>
</tr>
<tr>
<td>9</td>
<td>2-13</td>
<td>1502.900</td>
<td>1502.900</td>
<td>2-16-3-5-7-9-10-12-13-15</td>
</tr>
<tr>
<td>10</td>
<td>2-13</td>
<td>815.124</td>
<td>815.120</td>
<td>2-16-3-5-7-9-11-13-15</td>
</tr>
<tr>
<td>11</td>
<td>2-13</td>
<td>823.701</td>
<td>823.710</td>
<td>2-16-3-5-8-10-12-13-15</td>
</tr>
<tr>
<td>12</td>
<td>2-13</td>
<td>446.748</td>
<td>446.740</td>
<td>2-16-3-5-8-11-13-15</td>
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<tr>
<td>13</td>
<td>2-13</td>
<td>84.434</td>
<td>84.434</td>
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<td>14</td>
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<td>45.794</td>
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<td>2-16-4-7-9-11-13-15</td>
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<tr>
<td>15</td>
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<td>46.276</td>
<td>46.276</td>
<td>2-16-4-8-10-12-13-15</td>
</tr>
<tr>
<td>16</td>
<td>2-13</td>
<td>25.099</td>
<td>25.098</td>
<td>2-16-4-8-11-13-15</td>
</tr>
</tbody>
</table>

The UE link flows corresponding to the base case and the three scenarios computed using the SPSA are illustrated in Table 4. The UE path flows corresponding to each scenario were computed using the EWUE and MEUE methods. Table 5 shows the EWUE and MEUE path flows for the base case as well as the paths as sequences of link numbers. The comparison of the EWUE and MEUE path flows reveals that
they are very similar for the base case and can be assumed to be identical for all practical purposes. The numerical results for the three perturbed scenarios (1 to 3) are presented in Table 6. They suggest that the EWUE and MEUE path flows are similar except for the paths having smaller flows in Scenarios 3, for which the percentage differences between the EWUE and MEUE path flows are relatively higher. This difference may be due to the precision error of computation. The closeness of the EWUE and MEUE path flows is also evident from the root mean square error (RMSE) between the EWUE and MEUE path flow vectors, reported for different scenarios in Tables 5 and 6. The UE link flow vector from Table 4 indicates that in Scenarios 1 and 2, link 4 carries no flow. Therefore, in these two scenarios, flows of paths labeled 13-16 should be zero (as these paths traverse through link 4; see Table 5). The expected flows of paths labeled 13-16 computed by the EWUE method are zero. By contrast, even after convergence the MEUE method results into some residual flow on paths labeled 13-16 in both Scenarios 1 and 2, as shown in Table 6. This residual flow may also be due to the precision error of computation.

Table 6

<table>
<thead>
<tr>
<th>Path number</th>
<th>Scenario 1</th>
<th>Scenario 2</th>
<th>Scenario 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>EWUE flow</td>
<td>MEUE flow</td>
<td>EWUE flow</td>
</tr>
<tr>
<td>1</td>
<td>499.152</td>
<td>499.150</td>
<td>1704.500</td>
</tr>
<tr>
<td>2</td>
<td>270.720</td>
<td>270.720</td>
<td>924.461</td>
</tr>
<tr>
<td>3</td>
<td>1798.400</td>
<td>1798.400</td>
<td>3068.070</td>
</tr>
<tr>
<td>4</td>
<td>975.384</td>
<td>975.390</td>
<td>1664.020</td>
</tr>
<tr>
<td>5</td>
<td>944.228</td>
<td>944.230</td>
<td>1704.500</td>
</tr>
<tr>
<td>6</td>
<td>512.112</td>
<td>512.110</td>
<td>924.461</td>
</tr>
<tr>
<td>7</td>
<td>499.152</td>
<td>499.150</td>
<td>1.706</td>
</tr>
<tr>
<td>8</td>
<td>270.720</td>
<td>270.720</td>
<td>0.925</td>
</tr>
<tr>
<td>9</td>
<td>1798.400</td>
<td>1798.400</td>
<td>3.071</td>
</tr>
<tr>
<td>10</td>
<td>975.384</td>
<td>975.390</td>
<td>1.666</td>
</tr>
<tr>
<td>11</td>
<td>944.228</td>
<td>944.230</td>
<td>1.706</td>
</tr>
<tr>
<td>12</td>
<td>512.112</td>
<td>512.110</td>
<td>0.925</td>
</tr>
<tr>
<td>13</td>
<td>0</td>
<td>7.96E-09</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>0</td>
<td>7.96E-09</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>7.96E-09</td>
<td>0</td>
</tr>
<tr>
<td>16</td>
<td>0</td>
<td>7.96E-09</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>RMSE 0.002</td>
<td>RMSE 0.009</td>
<td>RMSE 0.064</td>
</tr>
</tbody>
</table>

The EWUE solution characteristics for the Anaheim, Chicago Sketch and Chicago Regional networks are presented in Table 7. A common EWUE solution characteristic of the three networks is that the average number of paths per O-D pair for the three networks is in general small. This is because many O-D pairs have just one path in the EWUE set. The average number of EWUE paths for the Anaheim and Chicago Sketch networks are similar and relatively smaller than that of the Chicago Regional network. This can be attributed to the denser link topology of the Chicago Regional network. The size of the largest O-D path set in the EWUE solution varies considerably among the three networks, ranging from 6 for the Anaheim network to 192 for the Chicago Regional network. Table 7 illustrates that the majority of the paths in the UE set obtained by SPSA after the filtering process (explained in Section 5) are retained in the EWUE path set. The EWUE model excludes only those paths (by allocating zero flows) that traverse links with zero flows. This characteristic of the EWUE model can be observed from the numerical results for network 3 (see Scenarios 1 and 2). Table 7 also shows the computational time (in seconds) for obtaining the EWUE solution; it also includes the computational time required for the parameter calibration. The computational times for obtaining the EWUE solutions for the three networks are small and illustrate that the proposed EWUE method is a viable option in practice.

The comparison of the numerical results for network 3 indicates that the EWUE path flow solution may
be similar to that of MEUE for real networks. Two possible reasons for this are as follows. First, the entropy of the MEUE macro state (flow condition) dominates the other macro states. Second, the other possible macro states are symmetrically distributed around the MEUE macro state. The numerical experiments reveal three key issues of the MEUE model. First, it requires a good initial feasible solution which may play a significant role in deciding the rate of convergence. Second, the MEUE solution may not be in the feasible space \( (H) \) at lower levels of convergence. Third, even a converged MEUE solution may retain some solution noise. In contrast, the proposed EWUE model does not require an initial feasible solution, requires reasonably small computational time and is simple to implement. Therefore, EWUE can act as a potential alternative to MEUE for finding a single representative UE path flows in practice.

### Table 7

Characteristics of the EWUE solution for three real-scale test networks.

<table>
<thead>
<tr>
<th>Network</th>
<th>Number of paths in UE set by SPSA</th>
<th>Number of EWUE paths with non-zero flows</th>
<th>Average number of EWUE paths per O-D pair</th>
<th>Highest number of EWUE paths for an O-D pair</th>
<th>Computational time for EWUE including parameter calibration (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anaheim</td>
<td>1,719</td>
<td>1,710</td>
<td>1.216</td>
<td>6</td>
<td>0.08</td>
</tr>
<tr>
<td>Chicago Sketch</td>
<td>126,192</td>
<td>126,149</td>
<td>1.354</td>
<td>12</td>
<td>1.01</td>
</tr>
<tr>
<td>Chicago Regional</td>
<td>11,472,785</td>
<td>11,445,708</td>
<td>3.613</td>
<td>192</td>
<td>386.80</td>
</tr>
</tbody>
</table>

### 7. Concluding comments

The non-uniqueness of path flow solution is a known limitation of the UETAP formulation. Multiple path flow solutions can be obtained for a UETAP based on the solution method and the parameters used. This raises the question of the meaningfulness of a UETAP path flow solution for practical applications. This study seeks to determine a single representative UETAP path flow solution.

Past efforts have sought to identify a single representative UETAP path flow solution using the MEUE method based on the entropy maximization concept. This study proposes the EWUE as an alternate method for the determination of a single representative path flow solution for the UETAP based on the method of mean values from statistical thermodynamics. Under EWUE, the single representative path flow solution is obtained as the entropy weighted average of all possible path flow solution vectors for the UETAP. The EWUE path flow solution vector has the minimum Euclidean distance from all other solution vectors in the UETAP solution space. The mathematical model of the EWUE extends the proportionality condition proposed by Bar-Gera and Boyce (1999) for the PASs to the paths. The results of numerical experiments suggest that the path flow solutions under EWUE and MEUE are about identical. However, EWUE is simpler to implement than MEUE in practice. In contrast to the MEUE method, it does not require an initial feasible solution. In addition, as the EWUE model parameters are calibrated to satisfy the set of constraints, the EWUE path flow vector lies in the feasible set. Further, it entails a small computational effort for finding the solution. Hence, the simplicity of the implementation of EWUE represents its key benefit over MEUE. Therefore, the proposed EWUE method can act as a potential alternative to the MEUE method in practice.

The proposed EWUE method requires a precise set of UE paths. This study uses an approximate method for generating the set of UE paths. A future study direction will be to explore the development of an efficient UE path set generation process. Another interesting future direction is the possibility of linearizing the calibration equations by using logarithms to further increase the efficiency of the EWUE method.
model parameter calibration procedure.

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References


