A Day-To-Day Dynamical Model for the Evolution of Path Flows under Disequilibrium of Traffic Networks with Fixed Demand

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\textbf{A B S T R A C T}

Transportation networks are often subjected to perturbed conditions leading to traffic disequilibrium. Under such conditions, the traffic evolution is typically modeled as a dynamical system that captures the aggregated effect of paths-shifts by drivers over time. This paper proposes a day-to-day (DTD) dynamical model that bridges two important gaps in the literature. First, existing DTD models generally consider current path flows and costs, but do not factor the sensitivity of path costs to flow. The proposed DTD model simultaneously captures all three factors in modeling the flow shift by drivers. As a driver can potentially perceive the sensitivity of path costs with the congestion level based on past experience, incorporating this factor can enhance real-world consistency. In addition, it smoothens the time trajectory of path flows, a desirable property for practice where the iterative solution procedure is typically terminated at an arbitrary point due to computational time constraints. Second, the study provides a criterion to classify paths for an origin-destination pair into two subsets under traffic disequilibrium: expensive paths and attractive paths. This facilitates flow shifts from the set of expensive paths to the set of attractive paths, enabling a higher degree of freedom in modeling flow shift compared to that of shifting flows only to the shortest path, which is behaviorally restrictive. In addition, consistent with the real-world driver behavior, it also helps to preclude flow shifts among expensive paths. Improved behavioral consistency can lead to more meaningful path/link time-dependent flow profiles for developing effective dynamic traffic management strategies for practice. The proposed DTD model is formulated as the dynamical system by drawing insights from micro-economic theory. The stability of the model and existence of its stationary point are theoretically proven. Results from computational experiments validate its modeling properties and illustrate its benefits relative to existing DTD dynamical models.

\textbf{Key words:} Day-to-day dynamics, stability, path-shift behavior

\textbf{1. Introduction}

Transportation networks often face perturbed conditions due to diverse events such as traffic accidents, construction activities, work zones, and the opening of a new traffic link. The temporal extent of the network-related impacts of these perturbations can range from a few hours to several months. Under such conditions, some drivers find themselves on costlier paths compared to others. This disequilibrium of traffic motivates drivers to shift paths. Drivers on costlier paths shift to cheaper paths to reduce their individual travel cost, resulting in the evolution of traffic over time. Finally, a stationary state is reached in which there is no incentive to switch paths. This aspect raises an important modeling problem: how to represent the path-shift behavior of drivers under the disequilibrium of traffic networks? It has important implications for practice in terms of influencing the development of more effective
strategies to mitigate/calm the disruptive effects of perturbation events. The problem is generally modeled as a dynamical system that captures the aggregated effect of path shifts of drivers in the form of a rate of change of network flows (path flows or link flows) over time.

The modeling of path-shift behavior of drivers depends on the underlying event that leads to the disequilibrium of traffic network. These events can be divided into four categories based on their impact on network flows and the time extent of the impact. The most common among them is the traffic incidents or accidents that affect the network flows for a short duration. In general, such transient phenomena can be captured by within-day dynamics (Ben-Akiva et al., 1991; Friesz et al., 1989, 1993). The second category is the demand variation due to special events such as a football game or a concert; in general, such events also have short-term effects and can be captured by modeling within-day dynamics. The third category of events is construction activities and work zones, they lead to capacity reductions and full/partial blockage of some links. Depending on the extent of such events, their impacts can last from a few days to several months. The evolution of traffic under these events can be captured through the modeling of the day-to-day (DTD) dynamics. The fourth category of events is topological changes in the network such as a bridge collapse (Zhu et al., 2010; He and Liu, 2012), opening of a bypass to traffic, and the restriction of right-of-way from two-way to one-way. The cascading effects of such phenomena on traffic patterns may also last for a long time and can be captured using DTD dynamics. This research focuses on modeling the DTD dynamics under fixed travel demand.

DTD dynamical models can be classified into two categories based on their day-to-day adjustment process: deterministic process models and stochastic process models. Deterministic process models lead to a single (link/path) flow vector at each time step. Stochastic process models (Cascetta, 1989; Davis and Nihan, 1993; Cantarella and Cascetta, 1995; Watling and Hazelton, 2003) determine a probability distribution of possible flow vectors at each time step. Hence, while the equilibrium of a deterministic process model refers to a single flow vector, the equilibrium of a stochastic process model implies a probability distribution of possible flow vectors. Deterministic process models are characterized by multiple domains of attraction, and model output is dependent on the starting point. By contrast, stochastic process models have a single domain of attraction, and hence the final state is independent of the starting point. An additional advantage of a stochastic process model is its ability to capture the variability associated with the random parameters of the DTD evolution (Watling and Cantarella, 2013). However, a challenge for its application is the need to determine an initial probability distribution. Recently, Smith et al. (2014) propose models that exhibit characteristics of both deterministic and stochastic process models.

The deterministic process DTD models can be classified further into two categories (Watling, 1999): deterministic route choice DTD models and probabilistic route choice DTD models. Deterministic route choice DTD models (Smith, 1984; He et al., 2010) assume that perceived and measured travel times of paths are equal and lead to a Wardropian equilibrium which is typically non-unique in terms of path flows but unique in terms of link flows. In addition, a deterministic path choice based on bounded rationality can lead to boundedly rational user equilibrium. Probabilistic route choice DTD models (Horowitz, 1984; Cantarella, 1993; Cantarella and Cascetta, 1995) assume that perceived travel times of paths are different from their measured travel times and vary over individuals, leading to a stochastic user equilibrium. An attractive feature of probabilistic route choice DTD models is the property of unique equilibrium point. However, a key implementation challenge is the identification of the distribution
function and related parameters for the error term in the perceived path travel costs. By comparison, deterministic route choice models are easier to implement but can have multiple equilibrium points. The proposed study seeks to develop a deterministic process DTD model based on deterministic route choice that leads to a Wardropian equilibrium which is a widely used objective for long-term planning in practice.

In past studies, the DTD evolution of transportation network (link/path) flows has been modeled using both analytical and simulation-based approaches. Yang and Zhang (2009) classify the analytical approaches into five categories: the simplex gravity flow dynamics (Smith, 1983), the proportional-switch adjustment process (Smith, 1984; Smith and Winsten, 1995; Huang and Lam, 2002; Peeta and Yang, 2003), the tatonnement process (Friesz et al., 1994), evolutionary dynamics (Sandholm, 2001; Yang 2005) and the projected dynamical system (Zhang and Nagurney, 1995, 1996; Nagurney and Zhang, 1996, 1997). Several studies adopt simulation-based or field survey approaches to model the DTD dynamics (Mannering et al., 1990; Hu and Mahmassani, 1995; Mahmassani, and Stephan 1988; Caplice and Mahmassani, 1992; Mahmassani and Jou, 2000; Srinivasan and Mahmassani, 2000; Srinivasan and Guo, 2004). While these approaches consider several factors including the socioeconomic and weather-related variables to model the path-shift behavior, the analytical models typically rely on the path features for this purpose. There are three important path features that can influence the number of drivers shifting their path, namely, the current path cost, the current path flow and the sensitivity of the path cost to flow. To the best of the authors’ knowledge, there is no DTD model that simultaneously considers these three path characteristics to model the path shift behavior under traffic disequilibrium. In general, most studies consider only the path costs and current path flows to determine the flow shift proportions. In a real-world scenario, if several cheaper paths are available, a few drivers would likely shift to a path whose cost increases rapidly with flow compared to the other paths. Ignoring this factor in the modeling process implies that drivers do not consider the sensitivity of path cost to the flow, thereby raising issues of behavioral consistency. In addition, for flow update at each time step, the existing DTD models typically shift flow from all used paths to the single shortest path, or from the set of paths with costs above average to the set of paths with costs below average, or based on pair-wise comparisons of costs of used paths connecting an origin to a destination. But a more customized approach would be to identify a set of attractive paths for each origin-destination (O-D) pair based on the characteristics of the perturbation event, and then shift flows from other used paths to them while accounting for the subjectivity among drivers. However, the existing literature in this domain lacks criteria to identify meaningful attractive paths under traffic disequilibrium. This study seeks to address the aforementioned gaps in the literature and proposes a dynamical model that utilizes all three path characteristics identified heretofore to represent the DTD traffic dynamics. The model is based on the proportional-switch adjustment process and is derived by leveraging insights from the multi-commodity economic market.

Pioneering work in the DTD dynamical modeling domain is by Smith (1984) which assumes that drivers shift from a path to all other cheaper paths connecting the same O-D pair. Under this assumption, the time rates of change of flows are obtained by adding the pair-wise flow swaps for each path. A key advantage of this approach is that it allows the highest degree of freedom in modeling the path-shift process under the restriction that the direction of each swap vector is from the costlier path to the cheaper path. This restriction may be a reasonable perspective under rational driver behavior. However, there are two limitations of this approach. First, it allows flow swaps among the expensive paths with costs higher than other attractive paths for the O-D pair; this may not be consistent with real-world driver behavior.
Second, it relies on the pair-wise comparison of the path costs to determine the direction of flow shifts. Such pair-wise comparisons increase the computational effort, which may be significant as the number of paths increase.

Recently, He et al. (2010) proposed a link-based dynamical model based on the proportional-switch adjustment process. In this model, at each time step, a fraction of the flow is shifted from all expensive paths to the shortest path for each O-D pair. When the shortest path is non-unique, the best shortest path is decided by the inertia effect. Such an approach simplifies the computational process as the flow updates can be performed directly in the link space rather than path space, and obviates the need for storing paths as in the Frank-Wolfe algorithm (Frank and Wolfe, 1956). The limitation of this approach is that it restricts the flow shift from all used paths between an O-D pair to just one path (shortest path) even if multiple attractive paths exist. Therefore, it has the lowest degree of freedom in modeling the path shift process.

The modeling approach proposed in this study lies conceptually between the two extreme cases by Smith (1984) and He et al. (2010) in terms of the degrees of freedom. In it, flows are shifted from the set of expensive paths to the set of attractive paths for each O-D pair. The path set for an O-D pair is divided into sets of expensive and attractive paths using a threshold parameter. This approach allows for the variability in driver choice for shifting paths under the availability of multiple attractive paths but precludes flow shifts within expensive or attractive path sets. To account for the inertia effect, it shifts only a fraction of flows on expensive paths to attractive paths and prohibits flow shifts from the attractive paths. Thereby, in the proposed model, the inertia effect is dominant for drivers on the attractive paths. Its behavioral assumption is that drivers become aware of potentially attractive paths but are unlikely to identify the cheapest path. Hence, when drivers shift from expensive paths, they are likely to shift to a set of potentially attractive paths rather than just to the cheapest path. This behavioral perspective is motivated by Smith and Mounce (2011) and Kumar and Peeta (2014), and the proposed DTD model can be viewed as the generalization of the static traffic assignment model by Kumar and Peeta (2014).

Existing studies have shown that analytical DTD models are asymptotically stable (Smith, 1984, Zhang and Nagurney, 1996; Han and Du, 2012) and the steady state solution of these models coincides with the Wardropian user equilibrium (UE) solution (Zhang et al. 2001, Peeta and Yang, 2003). These properties of the DTD dynamical models are desirable for their applicability in practice, and are established for the proposed DTD model in Sections 4 and 5 of this paper. This raises an important question about the need for complex DTD dynamical models given that the UE solutions can be obtained using traffic assignment models. The reason for adopting the DTD dynamical approach is that it provides flexibility in incorporating travel choice adjustment rules (Watling, 1999; Watling and Hazelton, 2003) and can provide the time trajectories of the day-to-day traffic evolution. By contrast, the focus of the traffic assignment models historically has been on the traffic equilibrium and not on the trajectory by which this state is obtained (Friesz et al., 1994). As emphasized by Friesz et al. (1994), sometimes the time stream of impacts produced by the evolving traffic during the state of disequilibria may be of greater interest than the final steady state of traffic. This is because future benefits are discounted more than the short-term costs according to the standard principle of economic evaluation of transportation infrastructure improvement projects (Sinha and Labi, 2007). Therefore, in some cases, the sum of the costs of impacts accrued during the states of disequilibria induced by a transportation intervention project may outweigh the benefits that are likely to accrue in the future. This justifies the need for dynamical
DTD models, and for more behaviorally consistent approaches to represent the associated path shift evolution.

Models for traffic evolution under the DTD dynamics of a traffic network can be grouped into two categories based on their solution space: path-based models and link-based models. The distinction between them arises due to the non-uniqueness of the equilibrium point of path-based Wardropian DTD models, as discussed earlier. The proposed model belongs to the class of path-based models. There are multiple reasons for adopting a path-based approach. The path-based traffic evolution can be more readily used for planning and management under perturbation events compared to link-based traffic evolution. One example is the evaluation of traffic diversion plans before fully/partly blocking a link for construction. The path flow evolution can also provide some information on traffic origins and the paths by which traffic arrives to a potentially congested area. Such knowledge can ensure that traffic is potentially diverted much before it enters a congested area. Further, path-based traffic evolution can be useful for medium- to long-term location/relocation decisions of businesses. Path-based traffic flow evolution can also support the planning for the mobilization of resources during the construction period. It can also be used for designing path-based control strategies for operations such as signal coordination design that optimizes the through-flow. In addition, a link-based model does not address the issue of the path-choice behavior realism. For example, DTD traffic evolution moving towards the user equilibrium or system optimal conditions can be specified by models operating in the space of link flows and link costs but the link flow and link cost vectors themselves have no behavioral interpretation. These factors motivate the adoption of a path-based approach for modeling the DTD dynamics here.

He et al. (2010) list two important shortcomings of the path-based approach. First, path-based dynamical models require an initial path flow pattern, which is difficult to obtain or observe in practice. Technologically, the paths of all vehicles in the network can potentially be tracked using cell phone (Wang et al., 2010), GPS, RFID, blue-tooth or license plate recognition, or combinations thereof. Therefore, as mentioned by He et al. (2010), this shortcoming can be viewed as a cost or privacy issue rather than a modeling issue. Second, path-based models ignore the path-overlapping effect in the flow-shift process. This can be overcome by splitting the path choice decisions of drivers to the node level and modifying the proposed model to represent the flow splits at each node, as suggested by Smith and Mounce (2011). In the proposed study, the path choice decisions are made based on the destination, which is more consistent with real-world decision-making. Also, in contrast to the He et al. (2010) proposition, drivers may be more willing to shift to an entirely new path that has no common links with the current path if they find sufficient incentive to do so.

There are two approaches for representing the dynamical system. In the first approach, the state variables are represented in continuous time and in the second, they are represented in discrete time. Accordingly, the dynamical models are classified as discrete-time and continuous-time models (Watling and Cantarella, 2013). Both the continuous-time (Smith, 1984; Friesz et al., 1994; Zhang and Nagurney, 1996; He et al., 2010) and the discrete-time (Horowitz, 1984; Cantarella and Cascetta, 1995; Bie and Lo 2010) approaches have been used to model the DTD dynamics. The representation of the state variables in continuous time is convenient to analyze model properties such as solution existence and stability. However, continuous-time models cannot be used for implementation when the closed-form integrals of the differential functions of the model cannot be obtained. By contrast, discrete-time models can be implemented in a straight-forward manner using numerical techniques such as Euler and Runge–Kutta
methods. In addition, as DTD path choice decisions have a day-to-day context, discrete-time models are consistent with the real world choice process.

Insights from dynamical systems models in other fields suggest that for computational convenience continuous-time models are typically approximated by discrete-time version that represents the continuous-time models at discrete instances of time (Stuart and Humphries, 1998). A limitation of this approach is that the discrete-time version may not converge to the same steady state solution as its continuous-time counterpart (Prüfer, 1985; Mickens, 1987; Yee et al., 1991); the anomaly between the two versions is typically governed by the discretization step size. However, there is a contrarian view that a discretized version of a continuous-time autonomous dynamical model can preserve the inherent dynamics. In particular, Mohamad and Gopalsamy (2000) in the context of neural networks show that the conditions required for the asymptotic convergence of the discrete-time version of the dynamical system to the equilibrium state are the same as those for its continuous-time counterpart. In addition, the anomaly between the continuous and discrete versions can be bridged by the appropriate selection of the discretization step size. These two factors provide the rationale for the several continuous-time DTD dynamical models proposed in the literature. This study first proposes a continuous-time model (in Section 2) to investigate the mathematical properties and then provides a discrete-time version (in Section 6) for practical implementation.

In summary, there are two key contributions of this paper. First, it incorporates the sensitivity of path costs with respect to flows in modeling the DTD dynamics of path shift behavior of drivers under traffic disequilibrium. This enables the model to differentiate between the paths that have nearly equal costs but different sensitivity of cost with respect to flow. The behavioral motivation for adopting this factor is that when there are multiple attractive paths under traffic disequilibrium, a path that consists of links with a larger capacity (for example, more number of lanes) is likely to attract a larger number of drivers. As a driver may potentially differentiate paths based on the perception of the sensitivity of path cost to flow based on past experience and the knowledge of the network, incorporating this factor in the model can lead to better consistency with the real-world. It also computationally smoothes the time trajectory of path flows by decreasing the flow oscillations between the paths of an O-D pair. This is a desirable property for practice where the iterative solution procedure is typically terminated, though not justifiably, at an arbitrary point due to computational time constraints. Second, the proposed modeling approach provides a criterion to classify the set of paths for an O-D pair into two subsets under traffic disequilibrium: the set of expensive paths and the set of attractive paths. This facilitates the flow shifts from the set of expensive paths to the set of attractive paths, thereby providing a higher degree of freedom compared to that of flow shifts to just the shortest path. In addition, it also helps to preclude flow shifts among expensive paths. The DTD dynamics of path-shift behavior of drivers under traffic disequilibrium is modeled as a dynamical system. The stability of the dynamical model and the existence of its stationary point are theoretically proven.

The remainder of the paper is organized as follows. Section 2 presents the notation and the derivation of the proposed DTD dynamical model. Section 3 provides the proof of solution existence of the model. Section 4 illustrates the equivalency of the stationary point of the DTD dynamical model with the Wardropian equilibrium. Section 5 provides the stability analysis of the proposed DTD model. Some implementation issues of the model are discussed in Section 6. Section 7 discusses computational experiments and insights from its results. Section 8 presents some concluding comments.
2. Formulation of the DTD dynamical model

This section provides the notation, defines the problem, and formulates the proposed DTD dynamical model based on insights from economic markets.

2.1 Notation and problem definition

Let the transportation network of interest be represented by a strongly connected directed graph denoted as $G(N, A)$, consisting of node set $N$ and link set $A$. Let $W$ be the set of O-D pairs and $K$ the set of used paths. The set of used paths that connects the O-D pair $w \in W$ is represented as $K^w$ and the time invariant trip demand for the O-D pair $w \in W$ as $d^w$. The flow on path $k$ of the O-D pair $w$ is represented as $f^w_k$ and the cost of traveling on that path is denoted as $c^w_k$. The flow and the travel cost on link $a \in A$ are represented as $x_a$ and $g_a$, respectively. Here, we consider no link interactions; $g_a$ is a monotonically increasing continuous function of $x_a$. $f^w$ is the vector of path flows for O-D pair $w$ and $f$ the vector of path flows for all O-D pairs. $c^w$ is the vector of path costs for O-D pair $w \in W$ and $c$ is the vector of path costs for all O-D pairs. $x$ is the vector of link flows and $g$ is the vector of link costs. The link-path incidence relationship is represented by the variable $\Delta_a^k$ which takes the value 1 if link $a$ is on path $k$, and 0 otherwise. The link-path incidence relationship for all links and used paths is denoted by the matrix $\Delta$.

This study assumes that the trip demand is time invariant. Based on the notation and definitions, the following properties hold:

\[ x = \Delta . f \]  
\[ c = \Delta^T . g \]  
\[ \sum_{k \in K^w} f^w_k = d^w \quad \forall w \in W \]  
\[ \dot{d}^w = \frac{d}{dt} (d^w) = 0 \quad \forall w \in W \]  

In Equation (4), $d^w$ is the time rate of change of trip demand for the O-D pair $w \in W$ which is zero based on the assumption of constant trip demand. Under the assumption of time-invariant trip demand and network supply conditions, the network flow pattern can attain the state of user equilibrium (UE) following Wardrop’s first principle (Wardrop, 1952). Under UE, used paths between an O-D pair have equal cost which is less than or equal to cost of any unused path. When the network is subjected to a perturbed condition, some drivers find themselves on costlier paths compared to others. This induces the path-shift by drivers. Drivers on costlier paths shift to cheaper paths and a state of equilibrium can potentially be achieved over time. To visualize this process (Fig. 1), consider an O-D pair $w$ that has three used paths. The equilibrium state under the unperturbed condition is represented by the dotted ellipses on the flow-cost diagram in Fig. 1. The numbers in the ellipses correspond to the three used paths. An intermediate state after the perturbed condition is represented by solid ellipses in Fig. 1. Under this state,
the cost of travel \((c_i)\) on path 1 is the minimum for this O-D pair and drivers on the other two paths experience higher costs. Under equilibrium, the cost of all used paths should be equal to the minimum cost of travel for the O-D pair. Hence, in the intermediate state paths 2 and 3 have excess costs; the excess cost \((E_k^w)\) of a path \(k \in K^w\) is defined as the difference of its cost from the minimum cost of travel for the O-D pair. That is, \(E_k^w = c_k^w - c_{\min}^w\), where \(c_{\min}^w\) is the minimum of the vector \(c^w\). This definition of excess cost is intuitive but deviates from some other definitions of excess cost used in the literature (Friesz et al., 1984; Li et al., 2012). This excess cost may induce the tendency to shift paths. Thereby, some drivers on paths 2 and 3 will shift their paths on the next day to decrease their excess cost. This path shift will decrease the flow on paths having excess cost, which in turn will decrease the excess cost of these paths. The path shifts by drivers results in the evolution of traffic flow over time, and leads to an equilibrium where the costs of all used paths become equal. This leads to the following modeling problem that forms the main objective of this paper:

“How to represent the time trajectory of path flows under the state of disequilibrium of traffic resulting from the DTD dynamics induced by the perturbed condition of the network?”

\[ E_i = c_i - c_{\min} \]

\[ E_2 = c_2 - c_{\min} \]

\[ E_3 = c_3 - c_{\min} \]

\(E_i\) is excess cost of path \(i\)

\(E_2\) is excess cost of path 2

\(E_3\) is excess cost of path 3

**Fig. 1. Excess cost of paths in the state of disequilibrium.**

### 2.2 Derivation of the model

The state of disequilibrium is an inherent property of competitive multi-commodity markets due to multi-player dynamics involved in it. This paper inherits the basic principle of microeconomic theory representing the competitive multi-commodity market and transforms it to represent the evolving states of traffic under the disequilibrium of the transportation network.

In an economic market, for a commodity \(i\) if the excess demand \(\xi_i\) is positive then the price \(q_i\) goes up and if the excess demand is negative then the price \(q_i\) goes down. The equilibrium is achieved when the excess demand of all commodities becomes zero. The disequilibrium of a multi-commodity market can be represented by the following dynamical system of differential equations with the constraint that price is positive (Arrow and Hurwicz, 1958; Morishima, 1996):
In the case of disequilibrium of a transportation network, if the excess cost of a path is positive then some drivers on it shift paths to decrease their travel costs, and that reduces the flow on that path and its excess cost. Equilibrium is reached when the excess costs of all paths become equal to zero. Hence, an analogy exists between an economic market and a transportation network, where the excess demand of an economic market is analogous to excess cost of a transportation network and the commodity price of an economic market is analogous to path flow of a transportation network.

In the case of disequilibrium of a transportation network, for the paths having positive excess costs and non-zero flows, the flows will decrease over time due to the shift of some drivers from these paths to others. Using the analogy from the economic market, the time rate of change (decrease) in flow is considered to be proportional to the excess cost. Therefore,

\[
\frac{dq_i}{dt} = \xi_i(q) \quad \forall i
\]  

Equation (5)

In Equation (6), \( \lambda^w \) is the non-negative proportionality constant and its value may vary across O-D pairs, and \( P^w \) is the set of paths with positive excess costs for the O-D pair \( w \). Equation (6) reflects two aspects from the real world perspective. First, higher the excess cost of a path, higher will be the shift of drivers from that path. Second, if there are two or more paths for an O-D pair that have equal excess cost, then their rates of change of flow will be in the same proportion as the number of drivers currently using these paths. Equation (6) also indicates that the excess cost is a function of the path flow vector \( f \).

There is a fundamental difference between an economic market and a transportation network that restricts the direct use of Equation (5) for a transportation network. While in an economic market excess demand can be either positive or negative, in a transportation network context excess cost cannot be negative. Therefore, the differential equation (5) is used here to represent the time rates of change of flows of paths with positive excess costs only.

The flows of paths having zero excess costs will increase due to drivers shifting from the paths with positive excess costs to these paths. Hence, the time rate of change of flow for a path with zero excess cost is assumed here to be a positively valued function \( \phi \). The value of \( \phi \) will be zero if the excess costs of all paths are zero or the flows of all paths having excess cost are zero for the O-D pair \( w \); else it will have some positive value. The rate of change of flow for a path with zero excess cost can be represented by Equations (7) and (8):

\[
\frac{df_{lw}^w}{dt} = \phi_{lw}^w(f) \quad ; \forall l \in \bar{P}^w, \forall w \in W
\]  

Equation (7)

\[
\phi_{lw}^w(f) = \begin{cases} 
0 & \text{if } \sum_{k \in P^w} E_k^w(f) f_k^w = 0 \\
> 0 & \text{if } \sum_{k \in P^w} E_k^w(f) f_k^w \neq 0 
\end{cases} \quad ; \forall l \in \bar{P}^w, \forall w \in W
\]  

Equation (8)
Here, in Equations (7) and (8) represents the set of attractive paths having zero excess cost for the O-D pair \( w \). From Equations (3) and (4) we have:

\[
\frac{d}{dt}(d^w) = \frac{d}{dt} \left( \sum_{k \in K} f_k \right) = \sum_{k \in K} \frac{df_k^w}{dt} = 0 \quad ; \forall w \in W
\]  

(9)

Since the set of paths having positive excess cost \( P^w \) and set of paths having zero excess cost \( \overline{P}^w \) for an O-D pair \( w \) complement each other, that is \( K^w = P^w \cup \overline{P}^w \), the summation in Equation (9) can be split into two groups:

\[
\sum_{k \in P^w} \frac{df_k^w}{dt} + \sum_{l \in \overline{P}^w} \frac{df_l^w}{dt} = 0 \quad ; \forall w \in W
\]  

(10)

Using Equations (6), (7), and (10) we obtain:

\[
\sum_{k \in P^w} -\lambda_k^w f_k^w E_k^w(f) + \sum_{l \in \overline{P}^w} \phi_l^w(f) = 0 \quad ; \forall w \in W
\]  

(11)

If there is a single path with zero excess cost between an O-D pair, then the summation sign in the second term of Equation (11) vanishes and the analytical expression for \( \phi_l^w(f) \) is obtained using this equation. But the possibility of multiple paths with zero excess cost cannot be ruled out. If there is more than one path between an O-D pair with zero excess cost, then additional information is needed to derive the analytical expression for \( \phi_l^w(f) \). At this point we make an additional assumption. This assumption is based on the driver’s perception of the sensitivity of the path cost to flow discussed in Section 1.

**Assumption 1:** The shift of flow to a path having zero excess cost is inversely proportional to the sensitivity of that path’s cost to flow.

The first derivative (slope) of the cost function of a path is used as the measure of sensitivity of that path’s cost to flow in this study. Using Assumption 1,

\[
\frac{df_l^w}{dt} \propto \frac{1}{s_l^w(f)} \quad ; \forall l \in \overline{P}^w, \forall w \in W
\]

\[
\Rightarrow \frac{df_l^w}{dt} = \phi_l^w(f) = \frac{\theta_l^w}{s_l^w(f)} \quad ; \forall l \in \overline{P}^w, \forall w \in W
\]  

(12)

In Equation (12) \( s_l^w(f) \) is the slope of cost function of path \( l \) and \( \theta_l^w \) is the constant of proportionality that varies across the O-D pairs. From Equations (11) and (12) we obtain:

\[
\sum_{l \in \overline{P}^w} \frac{\theta_l^w}{s_l^w(f)} = \sum_{k \in P^w} \lambda_k^w f_k^w E_k^w(f) \quad ; \forall w \in W
\]
Using Equations (12) and (13) the analytical expression for $\phi^w_i$ is as follows:

$$\phi^w_i(f) = \frac{\sum_{k \in P^w} \lambda^w f^w_k E^w_k(f)}{s^w_i \sum_{k \in P^w} \frac{1}{s^w_k(f)}} ; \forall l \in P^w, \forall w \in W$$

Using Equations (6), (7), and (14), the time trajectory of path flows under the state of disequilibrium of traffic resulting from the DTD dynamics under constant trip demand can be represented by the following dynamical system:

$$\dot{f}^w_i(t) = \frac{df^w_i}{dt} = \phi^w_i(f) ; \forall i \in \{k \in P^w, l \in P^w\}, \forall w \in W$$

where,

$$\begin{align*}
\phi^w_i(f) &= -\lambda^w f^w_k E^w_k(f) ; \forall k \in P^w, \forall w \in W \\
\phi^w_i(f) &= \frac{\sum_{k \in P^w} \lambda^w f^w_k E^w_k(f)}{s^w_i \sum_{k \in P^w} \frac{1}{s^w_k(f)}} ; \forall l \in P^w, \forall w \in W
\end{align*}$$

Equation (15) signifies that this study considers an autonomous system (for details see Khalil, 1996) and the time rate of change of path flows does not depend explicitly on time $t$.

2.3 Expanding the attractive path set in the model

The dynamical system represented by Equation (15) considers all three path characteristics identified earlier (the path cost, the current path flow, and the sensitivity of path cost to flow) in the determination of the flow shift between the paths of an O-D pair. If the shortest path is non-unique and there are multiple paths with zero excess cost, the slopes of cost functions of these paths are used to determine the proportions of flow shift to them. A drawback of this model is that it shifts flows from all the expensive paths (with positive excess cost) to just the shortest path(s) between an O-D pair. In the real-world context of traffic disequilibrium, drivers will not only shift from paths with positive excess cost to the minimum cost paths but also to other potentially attractive paths having costs close to the minimum path cost for the O-D pair. This is more likely to happen if these potentially attractive paths have higher capacity and their costs are less sensitive to flow. To visualize this situation, Fig. 2 illustrates a fourth path and modifies the flow-cost diagram for the O-D pair $w$ in Fig. 1 by adding a point corresponding to path 4. To simplify the representation, the points corresponding to the unperturbed
condition are omitted from this figure. Let path 4 has a very small excess cost as shown in Fig. 2. Then, some drivers from paths 2 and 3 may shift to path 4. Using the dynamical system represented by Equation (15), no flow will be shifted from paths 2 and 3 to path 4 but flow will be shifted from the path 4 to path 1. This is not consistent with real-world driver behavior. Hence, the attractive path set needs to be expanded by forming a variance band near the zero excess cost to include such potentially attractive paths. The variance band allows for the variability among drivers shifting paths in their perception of the attractiveness of alternate paths, by expanding the set of attractive paths beyond just the shortest path. The notion of a variance band differs from that of an indifference band (Mahmassani and Liu, 1999) which has been used to model the path shift decisions of a driver based on the cost difference between his/her current path and an alternate path under consideration. The upper bound ($\pi^w$) for the variance band is determined by a proximity parameter $\delta$ as shown in Equation (16):

$$\pi^w = \delta \cdot \max(E_i^w(f)), \quad i:\left\{ i \in P^* \text{and} f_i^w \neq 0 \right\}$$

(16)

The proximity parameter ($\delta$) can be interpreted as the drivers’ variance range in selecting alternative paths within which they treat paths alike. Based on this definition, $\delta$ should be small and less than 1. The paths that have excess costs ($E_i^w$) less than the threshold value $\pi^w$ will fall in this variance band and are labeled as attractive paths. The excess costs of attractive paths are adjusted to zero using Equation (17). Then, the dynamical system (15) entails flow shift to the potentially attractive paths (for example, to path 4 in Fig. 2) instead of flow shifts from these paths.

$$\begin{cases} E_i^w \to 0 & \text{if } E_i^w \leq \pi^w \\ E_i^w \to E_i^w & \text{if } E_i^w > \pi^w \end{cases}$$

(17)

Fig. 2. Expanding the cheaper set of paths for the state of disequilibrium.

The dynamical system (15) along with the conditions imposed by Equations (16) and (17) represents the DTD dynamical model proposed in this paper. As the excess costs are determined by the path flow vector, and the slopes of the cost functions are functions of the path flow vector, the dynamical system (15) can be represented in vector notation as $\dot{f}(t) = \phi(f)$. It is pertinent to note here that the
difference between the costs of expensive and attractive paths will diminish with time, and hence the variance band defined by Equation (16) will also shrink over time and tend towards zero. The shrinking variance band for attractive paths allows the stationary point of the dynamical system to converge to UE. Hence, the significance of the variance band decreases as the dynamical system moves towards the stationary point, and its role in separating expensive and attractive paths is more crucial in the early stages of disequilibrium.

3. Existence of solution of the DTD dynamical model

To establish the existence of the solution of the proposed DTD dynamical model, a property related to the domain and range of the dynamical system (15) is necessary. Hence, these properties are established first, and the proof of the solution existence follows thereafter.

Property 1. The domain $M$ of the function $\phi$ defined as $M = \{ f \in \mathbb{R}^{n} \mid f > 0 \text{ and } \sum_{k \in K^w} f_k^w = d^w \}$ is a positively invariant set with respect to the dynamical system represented by Equation (15) if the following condition holds for all time ($t$):

$$\lambda^w E_k^w(f) \leq 1; \forall k \in P^w, \forall w \in W$$

Proof. The set $M$ is said to be positively invariant in the dynamical system (15) if $f(0) \in M \Rightarrow f(t) \in M; \forall t \geq 0$. The sum of the rates of change of path flows for an O-D pair under the dynamical system (15) can be obtained as:

$$\sum_{k \in K^w} \frac{df_k^w}{dt} = \sum_{k \in K^w} -\lambda^w f_k^w E_k^w(f) + \sum_{l \in P^w} \lambda^w f_l^w E_l^w(f) \frac{1}{s_l^w(f)} \sum_{k \in P^w} \lambda^w f_k^w E_k^w(f) = 0$$

This result ($\sum_{k \in K^w} f_k^w(t) = 0$) implies that the quantity represented by the summation $\sum_{k \in K^w} f_k^w(t)$ does not vary with time. Hence, $\sum_{k \in K^w} f_k^w(0) = d^w \Rightarrow \sum_{k \in K^w} f_k^w(t) = d^w, \forall t \geq 0$.

Equation (17) also implies that $\dot{f}_k^w(t) \geq 0, \forall l \in P^w, \forall w \in W, \forall t \geq 0 \text{ if } f(0) \geq 0$. From Equations (17) and (20) we obtain, $\dot{f}_k^w(t) = -\left(\lambda^w E_k^w(f)\right) f_k^w \geq -f_k^w, \forall k \in P^w, \forall w \in W$. Hence, $f(0) \geq 0 \Rightarrow f(t) \geq 0, \forall k \in K^w, \forall w \in W$. This proves that $M$ is a positively invariant in the dynamical system (17).

Corollary 1. It follows from the proof of the property 1 that if $f(t) = \psi(f)$ is the solution of the dynamical system (15), then $\psi$ is a function from $M$ to itself, that is, $\psi : M \rightarrow M$. 

Corollary 2. The condition (18) defines the upper bound of the parameter $\lambda$. Recalling that $\lambda$ is a non-negative parameter, its bound can be defined as follows:

$$0 \leq \lambda^w \leq \frac{1}{\max(E^w(f))}; \forall w \in W$$

where $E^w(f)$ is the vector of excess costs of paths for the O-D pair $w$.

Property 2. (existence of fixed point solution) : Under the assumption that $g_a$ is a monotonically increasing continuous function and the existence of the lower bound $s_{k,w} > 0$ for the slopes of the path cost functions, the DTD dynamical model (15)-(17) has a fixed point solution.

Proof. Using the assumption that $g_a$ is a monotonically increasing continuous function and the slopes of the path cost functions $s_{k,w} \neq 0, \forall k, w$, it can be verified that $\phi^w(f)$ in Equation (15) is a continuous function. Hence, if $f(t) = \psi(f)$ is a solution of the dynamical system (15), then $\psi$ will also be a continuous function. Using corollary 1, $\psi$ is a function from $M$ to itself. We now make use of Brouwer’s fixed point theorem (Brouwer, 1910) which states that if $\psi$ satisfies these two conditions, it has a fixed point ($f^*$) in $M$ such that $\psi(f^*) = f^*$. It further implies that the dynamical system (15) has a stationary point in $M$ such that $\phi(f^*) = 0$; by definition this is an equilibrium point. Equations (16) and (17) do not affect these results as they simply expand the set $P^w$. This proves the existence of a stationary (equilibrium) point for the proposed DTD dynamical model (15)-(17).

4. Equivalency between the DTD dynamical model equilibrium and the Wardropian UE

The Wardropian UE is a widely-used objective to determine network flows for long-term planning. It is based on Wardrop’s first principle which states that UE is achieved when costs of all used paths between an O-D pair are equal, and less than or equal to costs of any unused paths (Wardrop, 1952). Thereby, each driver non-cooperatively seeks to minimize his/her cost of travel and UE is reached only when no driver can improve his/her travel cost by unilaterally switching paths. This is a reasonable representation of real-world network equilibrium under the assumption that all drivers behave rationally. Hence, it is imperative to prove that the stationary (equilibrium) point of the proposed DTD dynamical model coincides with the Wardropian UE. This property of the model can be stated as follows:

Property 3. The equilibrium of the DTD dynamical model (15)-(17) implies the Wardropian user equilibrium and vice versa.

Proof. Let the path flow vector $f^{UE}$ represent the Wardropian user equilibrium and the path flow vector $f^*$ represent the equilibrium of the DTD dynamical model (15)-(17). The Wardropian UE condition can be expressed as follows:
\[ f = f^{\text{UE}} \text{ if } \begin{cases} f_i^w > 0 \Rightarrow c_i^w = c_{\min}^w, & \forall i \in K^w, \forall w \in W \\ f_i^w = 0 \Rightarrow c_i^w \geq c_{\min}^w, & \forall i \in K^w, \forall w \in W \end{cases} \]

The equilibrium of the proposed DTD dynamical model is achieved when \( \phi = 0 \). Property 3 can be proved in two parts as follows:

Part 1: Wardropian UE implies equilibrium of the DTD dynamical model.

Let the Wardropian UE hold. When \( c_i^w = c_{\min}^w \), then \( E_i^w(f) = 0 \) and hence \( i \in \overline{P}^w \). When \( c_i^w > c_{\min}^w \), then \( f_i^w = 0 \); therefore \( \phi_i^w(f) = -\lambda^w f_i^w E_i^w(f) = 0 \) for all \( i \in P^w \). Therefore,
\[ \sum_{i \in P^w} -\phi_i^w(f) = \sum_{i \in P^w} \lambda^w f_i^w E_i^w(f) = 0, \forall w \in W. \]
Using Equation (15), we obtain
\[ \phi_i^w(f) = 0, \forall i \in \overline{P}^w, \forall w \in W. \]
Therefore, \( \phi^w(f) = 0, \forall w \in W \) or \( \phi(f) = 0 \) and the equilibrium of the DTD dynamical model is reached. Hence, \( f^{\text{UE}} \) implies \( f^* \).

Part 2: Equilibrium of the DTD dynamical model implies the Wardropian UE.

Let the equilibrium condition of the DTD dynamical model hold, that is, \( \phi(f^*) = 0 \). This implies \( \phi_i^w(f) = 0, \forall i \in K^w, \forall w \in W \). This, along with Equation (15), implies that either \( E_i^w(f) = 0 \) or \( f_i^w = 0, \forall i \in K^w, \forall w \in W \). Using Equations (16) and (17), it further implies that the flow of all paths in the set \( P^w \) is zero \( \forall w \in W \). As the computation of parameter \( \pi^w \) excludes the excess costs of paths having zero flow (see Equation (16)), \( E_i^w(f) = 0, \forall i \in \overline{P}^w \) is possible only when \( c_i^w = c_{\min}^w, \forall i \in \overline{P}^w \). Hence, either \( f_i^w = 0 \) or, \( c_i^w = c_{\min}^w, \forall i \in K^w, \forall w \in W \). Therefore, the Wardropian UE is satisfied, and hence \( f^* \) implies \( f^{\text{UE}} \).

5. Stability analysis of the DTD dynamical model

Stability is an important property of a dynamical model for its applicability in practice. This section presents the stability analysis of the DTD dynamical model proposed in Section 2. A dynamical system is characterized as stable or asymptotically stable system with respect to its equilibrium/stationary point using the Lyapunov method (Khalil, 1996). A dynamical system is said to be stable if all solutions starting at points near to the equilibrium point stay nearby. A dynamical system is said to be asymptotically stable if all the solutions starting at points nearby the equilibrium point not only stay nearby but also tend towards the equilibrium point with time. Section 4 established that the stationary point of the proposed DTD dynamical model coincides with the Wardropian UE. As the path flow solution of the Wardropian UE is non-unique, the DTD dynamical model is characterized by multiple stationary points. The stability analysis of a dynamical system with multiple stationary points can be performed using LaSalle’s theorem (Khalil, 1996), and has been adopted in this paper using insights from Peeta and Yang (2003). This theorem can be stated as follows.
Theorem 1. (LaSalle’s theorem). Let a dynamical system be represented as \( \dot{f}(t) = \phi(f) \). Let \( D \in \mathbb{R}^n \) be the domain of \( \phi(f) \) and \( \Omega \subset D \) be the compact set that is positively invariant with respect to the dynamical system. Let \( V \) be a continuously differentiable scalar function satisfying \( \dot{V}(f) \leq 0 \) in \( \Omega \). Let \( H \) be the set of all points in \( \Omega \) for which \( \dot{V}(f) = 0 \) and \( Z \) be the largest invariant set in \( H \). Then, all solutions of the dynamical system starting in \( \Omega \) are bounded and converge to \( Z \) as \( t \to \infty \).

Proof. The proof of this theorem can be found in Khalil (1996) and has been omitted here.

To prove the stability of the proposed DTD dynamical model using LaSalle’s theorem, we need to find the subset of the domain \( M \) of \( \phi(f) \) that is a positively invariant compact set and a scalar function \( V(f) \) that satisfies the conditions specified in theorem 1. This is done using the following lemmas and a theorem.

Lemma 1. Let \( \Omega_\beta \) be defined as \( \Omega_\beta = \{ f \in \mathbb{R}^n \mid V(f) \leq \beta \} \) where \( V(f) \) is a scalar function. If \( V(f) \) is radially unbounded, then \( \Omega_\beta \) is bounded for all values of \( \beta > 0 \). The function \( V(f) \) is said to be radially unbounded if:

\[
V(f) \to \infty \text{ as } \|f\| \to \infty
\]  

(19)

Proof. Let \( \rho \in \mathbb{R}^n \) be the point such that \( \beta = V(\rho) \). The condition (19) implies that there exists a \( f \) such that \( V(f) > \beta \) whenever \( \|f\| > \|\rho\| \). Let \( \omega \in \mathbb{R}^n \) be one such point for which \( \alpha = V(\omega) > \beta \). Thus, \( \Omega_\beta \subset \Omega_\alpha \), which implies that \( \Omega_\beta \) is bounded for all values of \( \beta > 0 \). For more details about the proof of this lemma and boundedness properties see Khalil (1996).

Theorem 2. Let \( \phi_i \) be the element of the DTD dynamical model that represents the flow shift of the path \( i \). Let the cost of any path in the set \( P_w \) be greater than the cost of any path in the set \( \bar{P}_w \) for all O-D pair \( w \in W \). If \( \sum_{k \in p^w} \phi_k^w(f) + \sum_{i \in p^\bar{w}} \phi_i^\bar{w}(f) = 0 \) holds for all O-D pairs \( w \in W \), then the dot product of the path cost vector \( c \) and the vector \( \phi(f) \) is non-positive; that is, \( c^T \phi(f) \leq 0 \).

Proof. \( c^T \phi(f) \leq 0 \) is true if \( (c^w)^T \phi^w(f) \leq 0 \) is true for all O-D pairs \( w \in W \). Hence, it is sufficient to prove that \( (c^w)^T \phi^w(f) \leq 0 \) holds for an O-D pair \( w \in W \). From the given condition in theorem 2,

\[
\sum_{k \in p^w} \phi_k^w(f) = -\sum_{k \in p^\bar{w}} \phi_k^\bar{w}(f)
\]  

(20)

Let the vectors of path costs for the sets \( P_w \) and \( \bar{P}_w \) be \( c^{p,w} \) and \( c^{\bar{p},w} \) respectively. Let \( c_{\min}^{p,w} \) be the minimum of the vector \( c^{p,w} \) and \( c_{\max}^{\bar{p},w} \) be the maximum of the vector \( c^{\bar{p},w} \). Then,
The equality of (21) will hold when all elements of the vector $c^w$ are equal. Using equations (20) and (21) we obtain:

$$c_{\max}^{P,w} \leq c_{\min}^{P,w}$$  \hspace{1cm} (21)

Further, $c_l^{w} \leq c_{\max}^{P,w} \forall l \in \overline{p}^w$ and $c_k^{w} \geq c_{\min}^{P,w}, \forall k \in p^w$. This leads to the following two inequalities:

$$c_l^{w} \leq c_{\max}^{P,w} \forall l \in \overline{p}^w \implies \sum_{l \in \overline{p}^w} c_l^{w} \phi_l^w (f) \leq \sum_{l \in \overline{p}^w} c_{\max}^{P,w} \phi_l^w (f)$$  \hspace{1cm} (23)

$$c_k^{w} \geq c_{\min}^{P,w} \forall k \in p^w \implies \sum_{k \in p^w} c_k^{w} (-\phi_k^w (f)) \geq \sum_{k \in p^w} c_{\min}^{P,w} (-\phi_k^w (f))$$  \hspace{1cm} (24)

Using Equations (22)-(24), we obtain:

$$\sum_{l \in \overline{p}^w} c_l^{w} \phi_l^w (f) \leq \sum_{l \in \overline{p}^w} c_{\max}^{P,w} \phi_l^w (f) \leq -\sum_{k \in p^w} c_{\min}^{P,w} \phi_k^w (f) \leq \sum_{k \in p^w} c_k^{w} (-\phi_k^w (f))$$

$$\implies \sum_{l \in \overline{p}^w} c_l^{w} \phi_l^w (f) \leq -\sum_{k \in p^w} c_k^{w} \phi_k^w (f)$$

$$\implies \sum_{l \in \overline{p}^w} c_l^{w} \phi_l^w (f) + \sum_{k \in p^w} c_k^{w} \phi_k^w (f) \leq 0$$

$$\implies \sum_{k \in k^w} c_k^{w} \phi_k^w (f) = (c^w)^T \phi^w (f) \leq 0$$

This completes the proof.

**Lemma 2.** The equilibrium condition of the DTD dynamical model (15)-(17) holds if and only if $c^T \phi(f) = 0$. That is, the vector $f^*(t)$ at time $t$ is the stationary point of the dynamical system $\dot{f}(t) = \phi(f)$ under the condition imposed by Equations (16) and (17) if and only if $c^T \phi(f^*) = 0$.

**Proof.** Part 1: Equilibrium of the DTD dynamical model implies $c^T \phi(f) = 0$.

Equilibrium is reached when path flow vector $f^*(t)$ does not change with time. Hence, at equilibrium, the elements of the vector $\phi(f)$ should be zero. That is, $\phi(f) = 0$ at equilibrium. Hence, $c^T \phi(f) = 0$ at equilibrium.
Part 2: \( c^T \phi(f) = 0 \) implies equilibrium of the DTD dynamical model.

This is proved using the method of contradiction. Suppose, \( c^T \phi(f) = 0 \) does not imply equilibrium. While proving Theorem 2 it was shown that \( (c^w)^T \phi''(f) \leq 0, \forall w \in W \). That is, \( (c^w)^T \phi''(f) \) for an O-D pair is either negative or zero. Hence, \( c^T \phi(f) = 0 \) is possible only when \( (c^w)^T \phi''(f) = 0 \) is true for all O-D pairs \( w \in W \). Therefore, \( c^T \phi(f) = 0 \) implies:

\[
\sum_{k \in P^w} c_k^w \phi_k^w(f) + \sum_{l \in \overline{P}^w} c_l^w \phi_l^w(f) = 0, \forall w \in W
\]  

(25)

We also know that \( \sum_{k \in P^w} \phi_k^w(f) + \sum_{l \in \overline{P}^w} \phi_l^w(f) = 0 \). Multiplying this with \( c_{\text{min}}^w \), the minimum of the vector \( c^P^w \), we obtain:

\[
\sum_{k \in P^w} c_{\text{min}}^w \phi_k^w(f) + \sum_{l \in \overline{P}^w} c_{\text{min}}^w \phi_l^w(f) = 0, \forall w \in W
\]  

(26)

Subtracting Equation (26) from Equation (25), we obtain:

\[
\sum_{k \in P^w} (c_k^w - c_{\text{min}}^w) \phi_k^w(f) + \sum_{l \in \overline{P}^w} (c_l^w - c_{\text{min}}^w) \phi_l^w(f) = 0, \forall w \in W
\]  

(27)

In Equation (27), \( \phi_k^w(f) \leq 0 \) and \( (c_k^w - c_{\text{min}}^w) \geq 0 \) for all \( k \in P^w \). Hence, the product \( (c_k^w - c_{\text{min}}^w) \phi_k^w(f) \leq 0 \) for all \( k \in P^w \). Similarly, the product \( (c_l^w - c_{\text{min}}^w) \phi_l^w(f) \leq 0 \) because \( \phi_l^w(f) \geq 0 \) and \( (c_l^w - c_{\text{min}}^w) \leq 0 \) for all \( l \in \overline{P}^w \). Therefore, Equation (27) is valid only when each term of this equation is zero. This implies \( \sum_{l \in \overline{P}^w} (c_l^w - c_{\text{min}}^w) \phi_l^w(f) = 0, \forall w \in W \). However, \( c_l^w \) cannot be equal to \( c_{\text{min}}^w \) unless at equilibrium, where all elements of vector \( c^w \) are equal. Hence, \( \phi_l^w(f) = 0 \) for all \( l \in \overline{P}^w \).

Summing it over \( l \in \overline{P}^w \) we obtain \( \sum_{l \in \overline{P}^w} \phi_l^w(f) = 0 \). But, \( \sum_{k \in P^w} \phi_k^w(f) + \sum_{l \in \overline{P}^w} \phi_l^w(f) = 0 \), implying \( \sum_{k \in P^w} \phi_k^w(f) = 0 \). Since \( \phi_k^w(f) \leq 0 \) for all \( k \in P^w \), \( \sum_{k \in P^w} \phi_k^w(f) = 0 \) is possible only when \( \phi_k^w(f) = 0 \) for all \( k \in P^w \). Therefore, Equation (27) is valid only when \( \phi_k^w(f) = 0 \) for all \( k \in K^w \) and \( w \in W \). This contradicts the proposition that \( c^T \phi(f) = 0 \) does not imply equilibrium. Hence, \( c^T \phi(f) = 0 \) implies equilibrium of the DTD dynamical model.

**Theorem 3.** Let \( g_a(x_a), \forall a \in A \), be positively valued and continuously differentiable functions. Then, the proposed DTD dynamical model represented by (15)-(17) is asymptotically stable.
**Proof.** As mentioned earlier in this section, a scalar function that satisfies the conditions specified in theorem 1 is necessary for the stability analysis of the proposed DTD dynamical model. A scalar function that satisfies these conditions is called a Lyapunov function. The search of Lyapunov function is a difficult task as there is no specified rule for finding this function. Peeta and Yang (2003) suggest that Beckmann’s UE objective function can be a candidate Lyapunov function for the dynamical systems; it is used for the stability analysis of the proposed DTD dynamical model. The Beckmann’s UE objective function is as follows:

\[ V = \int_0^{x(t)} g(x)^T dx; \quad x = \Delta f \quad c = \Delta^T g \]  \hspace{1cm} (28)

where, \( g(x)^T \) represents the transpose of the vector of link costs and \( x(t) \) is vector of link flows which varies with time \( t \). The scalar function \( V \) can be parameterized by \( t \) using the diagonal rule as follows:

\[ V = \int_0^{x(t)} g(x)^T dx = \int_0^t g(x)^T \frac{dx}{dt} dt \]

Since \( x = \Delta f \), \( \frac{dx}{df} = \Delta \). This implies that \( g(x)^T \frac{dx}{df} = g(x)^T \Delta = (\Delta^T g(x))^T = c^T \). Also, \( \frac{df}{dt} = \dot{f}(t) = \phi(f) \). Therefore, the derivative of the scalar function \( V \) is:

\[ \dot{V}(f(t)) = \frac{dV}{dt} = c^T \phi(f) \]  \hspace{1cm} (29)

The domain \( M \) of the function \( \phi(f) \) is defined as \( M = \{ f \in \mathbb{R}^n | f > 0 \text{ and } \sum_{k \in K^*} f_k^w = d^w \} \).

As the link cost functions \( g_a(x_a) \) are positively valued and continuously differentiable \( \forall a \in A \), the scalar function \( V \) is positively valued and continuously differentiable in \( M \). It can be verified that \( V(f) \rightarrow \infty \) as \( \| f \| \rightarrow \infty \). Hence, the set \( M \) is bounded by lemma 1 and is a closed set by definition; therefore, it is a compact set. Using property 1, \( M \) is also a positively invariant set if \( \lambda \) is bounded as specified by Equation (18). Now, using theorem 2 and Equation (29), the derivative of the scalar function \( V \) can be specified as \( \dot{V}(f) = c^T \phi(f) \leq 0 \). Let the path flow vector \( f^* \) represent the stationary point of the DTD dynamical model (15)-(17). Then, using lemma 2, we obtain \( \dot{V}(f^*) = c^T \phi(f) = 0 \). As \( f^* \) is not unique, let \( F^* \) be the set of all equilibrium/stationary points. From property 2, the set \( F^* \) is non-empty. Further, all points in \( F^* \) are possible future states starting from a point in \( M \) which is an invariant set. Hence, \( F^* \subseteq M \) and \( F^* \) is the largest invariant set in \( F^* \) itself. Therefore, using LaSalle’s theorem all solution points in \( F^* \) are bounded and the DTD dynamical model (15)-(17) will converge to a point in \( F^* \) irrespective of the starting point \( f \) in \( M \). Thereby, all intermediate states in the proposed DTD
dynamical model are closer to the equilibrium point than all past states irrespective of the starting point. This proves the asymptotic stability of the DTD dynamical model (15)-(17).

6. Implementation issues

This section presents some practical aspects related to the proposed DTD dynamical model from an implementation perspective.

The DTD dynamical model (15)-(17) is derived by considering time as a continuous variable. However, to apply this model in practice, a discrete-time version of this model is needed. The discrete version of the model reflects the real-world process of repeated decision-making and adaptation of commuting travelers better than continuous-time DTD models (Watling and Cantarella, 2013). In addition, the discrete version of the model can be implemented in a straightforward manner to determine the DTD flow evolution from a given initial condition. This study adopts Heun’s method, also known as the improved Euler method (Zarowski, 2004), to obtain the discrete-time version of this model as follows:

$$\bar{f}_{k,t}^{w,t+1} = f_{k,t}^{w,t} + h \phi_k^w (f_{k,t}^{w,t})$$  \hspace{1cm} (30)

$$f_{k,t+1}^{w,t} = f_{k,t}^{w,t} + \frac{1}{2} \left[ h \phi_k^w (f_{k,t}^{w,t}) + h \phi_k^w (\bar{f}_{k,t}^{w,t+1}) \right]$$  \hspace{1cm} (31)

$$\left\{ \begin{array}{l} 
\phi_k^w (f) = -\lambda^w f_k^w E_k^w (f) \quad \forall k \in P^w, \forall w \in W \\
\phi_l^w (f) = \sum_{k \in P^w} \lambda^w f_k^w E_k^w (f) s_l^w \frac{1}{s_k^w (f)} \quad \forall l \in \bar{P}^w, \forall w \in W 
\end{array} \right. \hspace{1cm} (32)$$

where $P^w$ and $\bar{P}^w$ are defined by Equations (16) and (17), $h$ is the step size, $f_{k,t}^{w,t}$ and $f_{k,t}^{w,t+1}$ are the flows of path $k$ belonging to O-D pair $w$ on day $t$ and $t+1$, respectively. Equations (30)-(32) suggest that the computation of path flows will require the computation of the product of two parameters $h$ and $\lambda^w$. In addition to these two parameters, the model implementation will also require the value of the proximity parameter $\delta$. To decrease the number of parameters the product $h \lambda^w$ is replaced by $h \lambda_{\text{max}}^w$, or equivalently $\lambda_{\text{max}}^w$ in Equation (32) replaces $\lambda^w$ with the condition $0 < h \leq 1$. Here, $\lambda_{\text{max}}^w$ is defined by corollary 2 as $\lambda_{\text{max}}^w = 1 / \max(E^w (f))$. The other two parameters ($\delta$ and $h$) of the discrete-time version of the DTD dynamical model need to be calibrated using real-world data. Here, it is imperative to mention that the values of these two parameters will depend on the time step selected in the discrete version of the model. In this study, the time step is assumed to be one day. In the absence of real-world data, this study uses parametric sensitivity analysis to identify their optimal values for the study experiments as those that lead to a balance between a smoother path flow profile and the rate of convergence of the DTD dynamical model. A value of $\delta = 0.1$ is used based on this analysis. The optimal step size ($h$) varies with the model for the three test networks used in this study, and is determined through the sensitivity analysis.
Another implementation issue relates to the slope of path cost function \( s^w_k(f) \). The existence of a solution requires \( s^w_k(f) \neq 0, \forall k, w \) (see Section 3). This is ensured by imposing a lower bound \( s_{LB} \) for \( s^w_k(f) \) when the value of \( s^w_k(f) \) approaches zero. This study uses a value of \( 10^{-7} \) for \( s_{LB} \).

7. Numerical experiments

Numerical experiments are conducted to analyze the proposed DTD dynamical model using three test networks of different sizes accessed from the website “Transportation Network Test Problems” (Bargera). The characteristics of these real-world test networks are presented in Table 1; among them the Anaheim network is the smallest and the Chicago Regional network is the largest. However, the O-D demand matrix used for the Chicago Regional network is different from that of this website.

<table>
<thead>
<tr>
<th>Network</th>
<th>Nodes</th>
<th>Links</th>
<th>Zones</th>
<th>O-D pairs with non-zero demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anaheim</td>
<td>416</td>
<td>914</td>
<td>38</td>
<td>1,416</td>
</tr>
<tr>
<td>Chicago Sketch</td>
<td>933</td>
<td>2,950</td>
<td>387</td>
<td>93,135</td>
</tr>
<tr>
<td>Chicago Regional</td>
<td>12,982</td>
<td>39,018</td>
<td>1,790</td>
<td>3,168,206</td>
</tr>
</tbody>
</table>

The proposed DTD model, labeled DTD_KP, is compared with the DTD models proposed by Smith (1984) and He et al. (2010), labeled as DTD_S and DTD_HL, respectively. The DTD_HL used in the experiments is the equivalent path-based version of the discrete-time link-based model of He et al. (2010). In DTD_HL, at each step, flow is shifted from all used paths to the shortest path for the O-D pair. For all three models, at each time step, first the path set is updated for each O-D pair by adding the shortest path if it is not present in the current path set. After updating the path sets of all O-D pairs, flows are updated sequentially by O-D pair. This study uses Dijkstra's algorithm with priority queue for generating the shortest paths. In all three models the proportion of flow shifts on each day is determined by a constant step size. The step size is a measure of the inertia of drivers shifting paths; a smaller step size implies a higher inertia. As discussed in Section 6, the optimal step sizes for the three models for each test network are determined through sensitivity analysis. Thereby, the study uses step sizes of 0.2, 0.1 and 0.05 for DTD_KP, DTD_S and DTD_HL, respectively, for the Anaheim network, 0.05 for all three models for the Chicago Sketch network, and 0.025 for all three models for the Chicago Regional network. In addition, to illustrate the sensitivity of the DTD models with respect to this parameter, the computational results are also presented using a step size of 0.05 for the Chicago Regional network for all three models. The results of the sensitivity analysis of the two parameters (step size and proximity parameter) for the DTD_KP model are also presented for the Chicago Sketch network so as to derive relevant insights. The link costs are determined using BPR functions without considering toll and travel distance factors.

The discrete-time versions of the three models were coded in C++ using the same data structure and implemented on a high-performance computing cluster with 48 cores and 192 GB of memory per
node, where the nodes are interconnected through a 10 Gigabit Ethernet network. The path flow UE solution (for the unperturbed condition) was first obtained by running the discrete-time version of the DTD_KP using an all-or-nothing assignment as the initial solution for the test networks. The UE path flow vector served as the initial point for the subsequent execution of the three DTD models. An O-D pair with multiple used paths (paths with non-zero flow) was selected for each test network, as shown in Table 2. The free flow travel times for one link on each used path (shown in bold in Table 2) were increased by varying amounts and their flow capacities were reduced by varying amounts to simulate the creation of work zones and to induce disequilibrium in the network. The three DTD models were then executed to determine the path flow evolution for the various networks. Table 2 presents the path cost evolution for the selected O-D pairs for the Anaheim and Chicago Sketch networks obtained using DTD_KP for the first few days; the path cost evolutions of the other two models are not shown as they are similar. The path cost evolutions for the Chicago Regional network using the three models differ significantly and are presented separately along with the flow evolution.

Table 2
Paths for the O-D pairs for the test networks.

<table>
<thead>
<tr>
<th>Path No.</th>
<th>Sequence of links comprising the path</th>
<th>Used path costs (minutes) obtained using DTD_KP</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Day 0</td>
</tr>
<tr>
<td>Anaheim network</td>
<td>1 49 755 756 704 706 669 613</td>
<td>8.102</td>
</tr>
<tr>
<td></td>
<td>2 49 753 700 704 706 669 613</td>
<td>9.228</td>
</tr>
<tr>
<td></td>
<td>3 49 755 758 760 706 669 613</td>
<td>8.248</td>
</tr>
<tr>
<td></td>
<td>4 49 753 698 662 607 613 613</td>
<td>7.301</td>
</tr>
<tr>
<td></td>
<td>5 49 755 756 702 667 669 613</td>
<td>8.102</td>
</tr>
<tr>
<td></td>
<td>6 49 753 698 663 665 613 613</td>
<td>7.302</td>
</tr>
<tr>
<td></td>
<td>7 49 753 698 663 667 669 613</td>
<td>7.303</td>
</tr>
<tr>
<td>Chicago Sketch network</td>
<td>1 10 1044 1055 1072 791 1081</td>
<td>15.686</td>
</tr>
<tr>
<td></td>
<td>2 10 1044 1053 780 784 791 1081</td>
<td>18.743</td>
</tr>
<tr>
<td></td>
<td>3 10 1041 776 780 784 791 1081</td>
<td>15.247</td>
</tr>
<tr>
<td></td>
<td>4 10 1043 1051 1067 795 1080 1081</td>
<td>15.262</td>
</tr>
<tr>
<td></td>
<td>5 10 1043 1051 1067 792 791 1081</td>
<td>15.279</td>
</tr>
<tr>
<td>Chicago Regional network</td>
<td>1 1756 35437 25285 25281 23439 32426 32428 23531 23529 32438</td>
<td>23439</td>
</tr>
<tr>
<td></td>
<td>2 1756 35436 17523 25276 35736 25261 23547 23537 32428 23531 23529 32438</td>
<td>32438</td>
</tr>
<tr>
<td></td>
<td>3 1756 35437 25285 25281 23437 23547 23537 32428 23531 23529 32438</td>
<td>32438</td>
</tr>
<tr>
<td></td>
<td>4 1756 35438 25298 25291 23400 23405 23430 23428 23446 23497 19577 23505</td>
<td>32438</td>
</tr>
<tr>
<td></td>
<td>5 1756 35436 17523 25275 25285 25281 23437 23547 23537 32428 23531 23529 32438</td>
<td>32438</td>
</tr>
<tr>
<td></td>
<td>6 1756 35437 25285 25281 23436 23434 23432 32426 32428 23531 23529 32438</td>
<td>32438</td>
</tr>
</tbody>
</table>

The proximity of the day-to-day evolving flows to the Wardropian UE is measured using the relative gap (Rgap):

\[
Rgap = \frac{\text{Total excess cost of the network}}{\text{Total travel cost of the network}} = \frac{\sum_{w \in W} \sum_{k \in K^w} f_k^w E_k^w(f)}{\sum_{w \in W} \sum_{k \in K^w} f_k^w c_k^w}
\]

The path flow evolution for the selected O-D pair and the solution convergence of the three DTD models for the Anaheim network are presented in Fig. 3. A common trend that emerges from the three plots in Fig. 3 is that the flows of paths labeled 1, 2 and 3 gradually reduce while those of paths 4, 6 and 7...
eventually increase. However, there are significant differences in their evolution profile especially for path 5. While under DTD_KP and DTD_HL flow of path 5 is negligible, DTD_S shows rise and then fall of its flow. Under DTD_S, the flows of some expensive paths increase and then eventually decrease; this includes paths that have costs lower than that of the costliest path for the O-D pair. This is due to pairwise shift of flow in DTD_S. The path costs for the given O-D pair of the Anaheim network in Table 2 indicate that drivers are likely to shift to paths labeled 4, 6 and 7 but not to paths labeled 1 and 5. Because such path shifts among the expensive paths are less likely in the real-world, the path flow evolution suggested by DTD_S may pose limitations for use in practice for traffic management decision-making.

The path flow evolution under DTD_HL has significant oscillations. This is because flow is shifted only to the shortest path from all other paths at each time step though other attractive paths are available. In addition, the flow shifts between the paths is not guided by their cost difference. Therefore, a set of attractive paths for an O-D pair keep changing their designation as the shortest path from one day to the next, leading to flow oscillations between them (see results for Chicago Regional network, Fig. 5 and 6). By contrast, DTD_KP provides a smoother path flow evolution profile as shown in Fig. 3. Smooth convergence is preferable in practice from a strategic decision-making perspective as an oscillating profile can be more difficult to interpret.

![Path flow evolution under DTD_KP](image1)
![Path flow evolution under DTD_S](image2)
![Path flow evolution under DTD_HL](image3)
![Convergence of the models](image4)

**Fig. 3.** Path flow evolution and convergence of the three DTD models for the Anaheim network.
The path flow outputs of the three DTD models differ significantly not only in the intermediate stages but also near the stationary point. This is because the path flow equilibrium point is non-unique and the underlying flow shift mechanisms of the three DTD models may lead to different equilibrium points. For instance, DTD_KP differs from both DTD_HL and DTD_S in the sense that flow shifts are not restricted to only the shortest path (as in DTD_HL), and flows do not shift to relatively expensive paths (as in DTD_S). While the usefulness of these models depends on their ability to replicate flow profiles observed using real-world data, the path flows from DTD_KP may be more intuitive for the aforementioned reasons.

Fig. 4 illustrates the path flow evolution for the selected O-D pair and the solution convergence under the three DTD models for the Chicago Sketch network. It depicts a similar trend for the path flow evolution profile as in Fig. 3. However, there is a significant difference in terms of the number of days required by the models to attain the stationary flows. In the Anaheim network (Fig. 3), DTD_S takes fewer days to obtain stationary flows than DTD_KP. For the Chicago Sketch network (Fig. 4), DTD_KP takes fewer days to obtain stationary flows than DTD_S. The comparisons of the path flow evolution and convergence plots of Figs. 3 and 4 suggest that the relative rates of convergence of the three models may vary from network to network but their path flow evolution characteristics remain consistent.
Figs. 5 and 6 illustrate the path flow evolution under the three DTD models for the selected O-D pair for the Chicago Regional network corresponding to the step sizes 0.025 and 0.05, respectively. The flow evolution profiles for this network (Figs. 5 and 6) portray a trend similar to the flow evolution profiles for the smaller networks (Fig. 3 and 4). However, for this network, the differences in the path cost evolution profiles of the three models were found to be more prominent than those of other two networks. Under DTD_KP, the excess costs of paths gradually approach zero (except for path 4 which has no flow). For DTD_S, the excess costs of all paths except path 2 (which has the highest cost) increase in the initial few days and then gradually decrease. This pattern is observed even for path 4 which has a very small flow (see Figs. 5(c) and 5(d)). The path cost evolution under DTD_HL shows oscillations which become more prominent at a larger step size, as illustrated by Fig. 6. As discussed earlier, this is due to the lower degree of freedom in the flow update process of DTD_HL. Oscillations, although of smaller magnitude, are also possible in flow and cost evolutions of DTD_KP under smaller values of the proximity parameter and larger values of the step size. Under such scenarios, a subset of paths can keep changing their designation between the expensive and attractive sets across days. In the limiting case when the proximity parameter is zero, the path flow and path cost evolutions under DTD_KP may show oscillations similar to DTD_HL. These oscillations are less prominent in the path cost and path flow evolution profiles of DTD_S due to higher degree of freedom in its path flow update process.
Fig. 5. Path flow and path cost evolution of the three DTD models for the Chicago Regional network using step size 0.025.

(a) Path flow evolution under DTD_KP
(b) Path cost evolution under DTD_KP
(c) Path flow evolution under DTD_S
(d) Path cost evolution under DTD_S
(e) Path flow evolution under DTD_HL
(f) Path cost evolution under DTD_HL
Fig. 6. Path flow and path cost evolutions of the three DTD models for Chicago Regional network using step size 0.05.

Fig. 7. Convergence of the three DTD models for the Chicago Regional network.

Fig. 7 shows the $Rgap$ convergence of the three models over time for the Chicago Regional network using the two values of step size. While smoother flow profiles were obtained using the step size 0.025, faster convergence was achieved for DTD_KP and DTD_HL using the step size 0.05. In terms of convergence, DTD_KP performs better than the other two models using step size 0.05. However, under the step size 0.025, DTD_S performs better than DTD_KP. The convergence of the three models for the Anaheim and Chicago Sketch networks are illustrated in Figs. 3(d) and 4(d). For these two networks, DTD_KP performs better than DTD_S which in turn performs better than DTD_HL.

The convergence plots (Figs. 3(d), 4(d) and (7)) show that DTD_S has a smoother convergence towards the Wardropian UE compared to other two models. DTD_HL in general shows an oscillating convergence characteristic. The magnitude of these oscillations increases with the increase in network size and step size for the same network. This can be attributed to the oscillating flow and cost profiles of DTD_HL. The oscillations for DTD_KP lie between those under DTD_HL and DTD_S. A common property that emerges from these plots is that the convergence rates of these models are sensitive to the step size. Hence, the calibration of this parameter is critical for the meaningful application of these DTD
models for practice. Table 3 shows the computational times of the three models to obtain the stationary flows for the test networks.

**Table 3**
Computational time required by the models for convergence.

<table>
<thead>
<tr>
<th>Network</th>
<th>DTD_KP</th>
<th>DTD_S</th>
<th>DTD_HL</th>
<th>Rgap</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anaheim</td>
<td>0.99 s (0.05)</td>
<td>0.84 s (0.05)</td>
<td>1.22 s (0.05)</td>
<td>1.0 E-04</td>
</tr>
<tr>
<td>Chicago Sketch</td>
<td>59.5 s (0.05)</td>
<td>99.87 s (0.05)</td>
<td>82.8 s (0.05)</td>
<td>1.0 E-05</td>
</tr>
<tr>
<td>Chicago Regional</td>
<td>175.8 m (0.025)</td>
<td>116.7 m (0.025)</td>
<td>228.6 m (0.025)</td>
<td>1.0 E-05</td>
</tr>
<tr>
<td>Chicago Regional</td>
<td>121.6 m (0.05)</td>
<td>449 m (0.05)</td>
<td>Not converged</td>
<td>1.0 E-05</td>
</tr>
</tbody>
</table>

*In Table 3, s indicates seconds, m indicates minutes, and the value in the parentheses indicates the step size used.

Fig. 8 illustrates the results of the sensitivity analysis of DTD_KP for two parameters (δ and h) for the Chicago Sketch network. The DTD_KP model was implemented for different values of these parameters, and the path flow evolution profiles were obtained. The left column in Fig. 8 shows the DTD path flow evolutions for different proximity parameter values (δ=0.05, 0.1, 0.2 and 0.3) for a fixed step size (h=0.05). They indicate that higher values of the proximity parameter may lead to the inclusion of some relatively expensive paths in the attractive path set leading to an increase in the flows of those paths (for example, path 1 in Fig. 8) in the initial days. The flows of these paths decrease in the later days. In this respect, DTD_KP shows a trend similar to DTD_S for higher values of the proximity parameter. The right column in Fig. 8 shows the DTD path flow evolutions for different values of the step size (h=0.025, 0.1, 0.3 and 0.6) under a fixed proximity parameter value (δ=0.1). They suggest that as the value of the step size increases, the oscillations in the DTD path flow evolution profiles increase. They also suggest that the number of days required to reach the equilibrium state varies significantly with the step size, and DTD_KP may fail to reach equilibrium for large values of the step size. For example, in Fig. 8, for h=0.6, the equilibrium is not attained in 60 days. The step size reflects the inertia of drivers on expensive paths; a larger step size implies a lower inertia. From a real-world perspective, if a large number of drivers keep shifting their paths, then the associated path flow evolution will display highly oscillating profiles. In the extreme case, when all drivers on expensive paths shift their path (h=1), the path flows may oscillate forever and equilibrium may not be achieved.
Fig. 8. Path flow evolution under DTD_KP with different parameter values for the Chicago Sketch network.

8. Concluding comments

Transportation networks are usually subjected to events that lead to disequilibrium of traffic. This induces path-shift by drivers that potentially evolves to a new equilibrium. This paper presents a dynamical system based formulation to model this path-shift behavior based on insights from economic markets, with the objective of further enhancing the behavioral underpinnings of such models for practice.

Current DTD dynamical models generally consider existing path flows and path costs, but neglect the sensitivity of path costs to flows, in modeling the path-shift process. This study bridges this gap in the literature and proposes a dynamical system based formulation that captures all three of these path characteristics. In addition, this study provides a criterion for selecting a set of attractive paths under traffic disequilibrium. The proposed DTD_KP lies conceptually between the two extremes of the DTD
models (Smith, 1984 and He et al., 2010). While DTD_S provides the highest degree of freedom in modeling and relies on pair-wise flow shifts between paths, DTD_HL addresses the inertia effect and allows flow shifts only to the shortest path at each step. Both models have potential behavioral limitations. While DTD_S results in shifting of flows among expensive paths, DTD_HL provides an oscillating flow profile due to shifts only to the shortest path. Hence, both have constraints from a practical application viewpoint. The flow shift to a set of expensive does not realistically reflect driver behavior in the real-world. An oscillating flow profile raises the question of confidence for its applicability in practice. The proposed DTD_KP model overcomes these limitations by shifting flows from the relatively expensive paths to set of attractive paths, while factoring inertia effects for the attractive paths. In addition, DTD_KP uses the sensitivity of path cost relative to flow in modeling the path-shifts to smoothen the flow evolution profile. These two steps help to improve the behavioral consistency in the DTD_KP. While all three models converge to the Wardropian UE, their path flow evolution and stationary points may be different. Consequently, as the underlying flow shift mechanism of DTD_KP is conceptually more consistent with the real-world driver behavior, its path flow evolution/solution may be more meaningful for developing effective traffic management strategies in practice.

The results of numerical experiments validate the modeling properties of the DTD_KP model and illustrate its comparative benefits. First, the path flow evolution profile obtained using DTD_KP does not show a flow increase for any expensive path at any time unlike under DTD_S which suggests flow increases for some relatively expensive paths in the initial days. Second, the path flow and cost evolution profiles under DTD_KP are smooth and do not show oscillations unlike under DTD_HL. Third, while all three models converge to the Wardropian UE in terms of link flows, their path flow profiles differ significantly, both at the intermediate stages and near the stationary point, after starting from the same initial point. This illustrates the potential value of adopting the behaviorally more consistent approach of DTD_KP for developing dynamic management strategies under the corresponding perturbations. Fourth, the convergence rates of the three models are sensitive to the step size. In the experiments of Section 7, except for one instance, DTD_KP converges faster than DTD_S which in turn converges faster than DTD_HL.

An ongoing extension of this study seeks to explore the potential convergence of the proposed DTD dynamical model to the boundedly rational user equilibrium by using a constant variance band. Another future direction is to calibrate the model parameters so that they can replicate real-world flow evolutions, and then compare the model outputs using real-world data.

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