Stability Issues for Dynamic Traffic Assignment

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Abstract

This paper explores stability issues for operational route guidance control strategies for vehicular traffic networks equipped with advanced information systems, and develops a general procedure for the stability analysis of the associated dynamic traffic assignment (DTA) problems. The route guidance control strategies are modeled as dynamical systems, and the associated solution procedure enables computational tractability for real-time deployment. An important study insight is that the Lyapunov functions for the route guidance control models are their corresponding objective functions under DTA. This overcomes the key difficulty of constructing meaningful Lyapunov functions for DTA problems.

Key words: Feedback control, dynamical systems, stability analysis, Lyapunov methods.

1. Background

Stability is an important operational issue for the control of dynamic vehicular traffic networks through information provision and route guidance. This is because inappropriate information dissemination or route guidance may lead to increased unpredictability and volatility, and/or catastrophic consequences for the traffic system, compromising the objectives of information provision. While well-defined properties cannot be assumed for general traffic networks given the intricacies and interactions associated with dynamic traffic phenomena (Peeta and Ziliaskopoulos, 2001), there are several key factors that characterize the complexity of dynamic traffic networks. They include incidents (supply conditions), randomness in time-dependent origin-destination (O-D) demand, driver behavior, dynamic variability in the composition of driver types in the traffic stream, information provision, driver response to the supplied information, and external traffic controls (such as signalization and ramp metering). Of these, external traffic controls and information provision represent deliberate intervention to positively influence system performance. Hence, understanding the effect of information provision on the stability of the traffic system is essential for the effective real-time operation of traffic networks equipped with advanced information systems.

Traffic assignment models aim to determine the network flow patterns for some time horizon of interest in terms of link/path flows that satisfy some pre-specified objectives given the O-D demands, the network structure, and link performance functions that relate flows to travel times in an aggregate manner. User equilibrium (UE) and system optimal (SO) are two common objectives used to determine network flow patterns (Wardrop, 1952). The UE principle states that traffic equilibrium is reached only when no driver can improve his/her travel time by unilaterally switching paths (routes). The SO principle states that the used paths and the associated flows are such that the total system travel time is minimized.

The traditional traffic assignment models (Sheffi, 1985) are static as they assume the O-D demands and network supply conditions to be time-invariant over the time horizon of interest, implying constant trip times and flows on network paths. Such assumptions are reasonable in the long-term planning context where the focus is on network design rather than on congested conditions. A stochastic formulation results when the drivers’ perceived travel times are treated as random variables to incorporate randomness in driver behavior. However, static assignment models do not suffice in an operational context where the time-dependency in demands, flows and travel times is of essence to the traffic system operation and control. This is especially so during peak-period traffic and under incidents. It manifests primarily as the need to adequately represent traffic flow realism so as to capture dynamic phenomena such as queue build-up and discharge, spillbacks, and variability in flow conditions.

The incorporation of time-dependency in terms of O-D demands, traffic flows and supply conditions leads to the dynamic traffic assignment (DTA) problem. A broad range of problems are addressed under the umbrella of DTA, each characterized by different decision variables, behavioral assumptions, and data needs, as well as solution objectives and functional requirements. Peeta (1994)
provides a comprehensive taxonomy of the various problem classes addressed under DTA. A conceptual extension of the static traffic assignment problem, DTA typically seeks the optimal time-dependent paths (or, more generally, path assignment proportions) and network flow patterns that satisfy some system-wide and/or user objectives. However, unlike its static assignment counterpart, the general DTA problem is characterized by ill-behaved properties, such as non-convexity and non-differentiability, arising from the need to realistically capture traffic dynamics and driver behavior. This leads to substantial complexities in the problem formulation, solution methodologies, and real-world implementation, and is the basis for the two primary approaches to address DTA problems: analytical and simulation-based. Properties such as existence and uniqueness, sought by analytical DTA approaches, can be guaranteed only through potentially limiting assumptions on driver behavior and/or traffic flow modeling, precluding generalization. Attempts to adequately represent traffic realism and driver behavior, which is the focus of simulation-based DTA solution approaches, preclude the guarantee of desirable mathematical properties. Hence, no single modeling approach provides a general solution, leading to trade-offs among various characteristics in the different models. A comprehensive review of DTA in terms of its characteristics, scope, issues and solution approaches is provided in Peeta and Ziliaskopoulos (2001).

Another manifestation of the complexity of the DTA problem is the dependence of the current assignment strategy on future traffic conditions. This entails the a priori knowledge of O-D demands and network supply conditions for the entire time horizon of interest so as to solve DTA problem, an unlikely proposition given the current advances in this field. The associated idealized problem assumes full knowledge of O-D demands and supply conditions for the entire time horizon of interest, and is called the deterministic DTA problem. While not a deployable formulation, it serves as a benchmark for the best performance achievable in that network through the assignment of vehicles to paths under the specific objective.

From an on-line deployment perspective, the problem intractability has led to a focus on developing computationally efficient solution procedures that are effective and robust, rather than emphasizing the guarantee of global optimality and uniqueness. That is, in deployable DTA approaches, the premium is on accurately representing traffic dynamics and ensuring on-line consistency with the unfolding traffic conditions. Due to the lack of satisfactory analytical models to replicate the traffic dynamics, simulation models are typically used to ensure traffic realism. The factors that characterize the competency of a deployable DTA approach are: solution effectiveness, solution robustness, stability, real-time computational tractability and on-line consistency.

A key application domain for DTA is in the context of real-time information provision to drivers through an advanced traffic management and information system. It entails the real-time processing of large amounts of time-dependent traffic data as part of the deployable DTA solution methodology. This places a hard constraint that the paths to be provided to drivers through route guidance be determined in sub-real time. In an effort to reduce the computational burden, a rolling-horizon methodology to deploy DTA was first proposed by Peeta and Mahmassani (1995a). It divides the deployment time horizon of interest, called the planning horizon, into several stages and sequentially solves a deterministic DTA problem for each stage at some time before the start of that stage. Thereby, in any stage, it circumvents the need to solve a DTA problem for the rest of the planning horizon. A practical advantage is the need for O-D demand predictions for only the near-term stage rather than the entire remaining time horizon. The solving of a DTA problem for the much smaller time duration represented by a stage reduces the computational time significantly. However, the typical deterministic DTA solution methodologies tend to be computationally intensive iterative search procedures, precluding the determination of solutions in sub-real time.

Hawas and Mahmassani (1995) propose the partitioning of the network into smaller zones, and apply reactive local heuristic rules within each zone to determine the path assignment proportions. The reactive nature of this strategy precludes the need for O-D demand and network supply predictions, and the use of local heuristic rules circumvents the need to solve a deterministic DTA problem in real-time. However, since historical data (on O-D demands and incidents) is not exploited and the solutions are localized, the strategy is not robust in general. Pavlis and Papageorgiou (1999) propose a reactive decentralized feedback control strategy to reduce the computational burden. It uses real-time traffic measurements to equalize the instantaneous travel times on the used paths for each O-D pair, leading to an instantaneous UE solution. It circumvents predictions and has a decentralized logic akin to Hawas and Mahmassani (1995). However, it also does not use historical data and its reactive logic does not formally incorporate the underlying driver behavior processes. Hence, its robustness is limited to specific network topologies. A common feature of these two deployable DTA approaches is the use of the instantaneous link/path travel times obtained through real-time traffic data measurements to determine the path assignments. It highlights an important difference between potential deployable DTA approaches and the idealized deterministic DTA solution which seeks the experienced link/path travel times. Experienced travel times are the actual travel times experienced by drivers unlike the instantaneous ones which are based on the current conditions. This implies that estimates of experienced travel times can be obtained only through the prediction of the future network states, entailing the solving of a deterministic DTA problem on-line as well as O-D demand and incident likelihood predictions. This introduces computational intractability issues and the need for robust
predictions. Even the rolling-horizon strategy, which solves deterministic DTA problems for smaller durations, is computationally intensive in a centralized architecture.

Peeta and Zhou (2002) propose a hybrid DTA deployment strategy that combines computationally intensive off-line and efficient on-line components. The off-line component solves deterministic DTA problems for several O-D demand realizations obtained from the historical O-D demand data. These solutions are used to generate a robust initial solution vis-à-vis randomness in O-D demand for on-line use. In addition, the deterministic DTA solutions for several likely incident scenarios, based on the historical incident data, are generated off-line. The on-line component consists of computationally efficient reactive heuristics that use the solutions generated off-line to update the initial solution based on the unfolding demand and supply conditions on a given day. This ensures that the historical data available for that traffic network is exploited while circumventing the need for on-line demand and supply predictions. Thereby, the time-dependent experienced path travel times under several likely O-D demand realizations are used to generate the initial on-line solution. Also, no DTA problem is solved on-line ensuring the sub-real time determination of the path assignment proportions.

Another key element in the context of information provision is the driver response to the supplied information. Most DTA formulations solve for the UE or SO objectives. The UE objective is popular because it is considered a reasonable representation of driver behavior, while SO serves as a benchmark for the best system performance and to design control strategies. Mahmassani et al. (1993) introduce a multiple user classes formulation that considers the information availability to the drivers, the information type, and driver response to information. A boundedly-rational rule is used as a representative driver behavior model; drivers switch to an alternative path en-route if travel time savings over the current route exceed a pre-specified threshold. While various other driver behavior models have been proposed in this context, deployable DTA models typically aim to provide the capabilities to incorporate any behavior model as driver behavior is likely to be network-dependent. Yet others assume pre-specified and/or dynamic compliance rates to replicate driver response using current traffic measurements or elaborate behavior models. However, most existing DTA formulations address the UE objective vis-à-vis driver behavior.

This paper explores stability issues for a proposed traffic assignment strategy that follows the feedback control framework shown in Fig. 1, under the UE and SO objectives. The traffic control center collects traffic information such as instantaneous link travel costs and flows from the field, and uses this information to provide guidance to drivers to improve real-time system performance. The study develops a general procedure for the stability analysis of dynamic traffic assignment problems. In this context, the notion of stability implies that: (1) all solutions are bounded and converge to time-dependent desirable states, and (2) the proposed control strategies minimize or limit the deterioration of system performance. While a guarantee of stability is unrealistic for general networks given the various factors that influence it, this study seeks insights on the effects of information provision. We show that the proposed strategy moves the system toward the desirable time-dependent stable states, should such states exist.

**Fig. 1 about here**

From a deployment perspective, the planning horizon is divided into small consecutive intervals, labeled assignment intervals, for which the path assignment proportions are to be determined. Towards the end of the current assignment interval, O-D demand estimates are assumed to be available for the next assignment interval. The real-time measurements of current traffic flows/costs are used in conjunction with the route guidance control strategy model to obtain a stable solution for the next assignment interval. This procedure is repeated until the end of the planning horizon.

The solution procedure consists of a dynamical system, a traffic simulator, and an algorithm to generate the instantaneous time-dependent link performance functions. The route guidance control strategy is modeled as a dynamical system. Let \( \tau \) denote the current assignment time interval in the traffic system. It represents the time period in which the stable path assignment proportions are to be determined for the next assignment interval. Towards the end of \( \tau \), real-time data from the traffic network in terms of link travel times and flows are used to generate the time-dependent link performance functions. Using these functions, the O-D demands for \( \tau +1 \), the vehicle locations obtained from the traffic simulator, and the route switching strategy discussed in Section 4, the dynamical system determines the time-dependent path assignment proportions for \( \tau +1 \). The dynamical system is a virtual traffic system in which \( \tau \) is fixed, enabling the use of static assignment concepts. Thereby, in the dynamical system, the O-D flows for \( \tau +1 \) are constant and all other vehicles in the network towards the end of \( \tau \) are background flows. This is conceptually illustrated in Fig. 2. Let \( t \) denote a time unit in the dynamical system. It does not correspond to physical time in the traffic system and only tracks the progress of the traffic network state in the dynamical system solution search procedure. It is analogous to the iteration number in an iterative search procedure. The dynamical system searches for the stable desirable states by switching flows across paths using travel costs updated through the time-dependent link performance functions for that \( \tau \). Thereby, while the O-D demands and link performance functions are fixed in the dynamical system, they vary across \( \tau \). This enables the approach to capture the time-dependency while ensuring computational tractability through the use of the dynamical system. The \( y \)-axis in Fig. 2 represents the progress of time in the physical world (the traffic network) and the \( x \)-axis represents the virtual time in the dynamical system. Each layer along the \( x \)-axis corresponds to the updated solution along the time trajectory \( t \) of the dynamical system. At
each assignment interval $\tau$, there is a set of layers along the $x$-axis that indicate the progress of the solution search process in the dynamical system. The dynamical system search is terminated at $t_f$ when an acceptable solution is achieved for assignment interval $\tau + 1$. The solid and dotted lines in the virtual network represent different path assignment proportions in the search process. By adjusting the path assignment proportions using the Runge-Kutta method, the dynamical system searches for the desirable DTA solution defined by the prescribed objectives. The associated path assignment proportions are inputs to the traffic simulator which then replicates the traffic conditions for $\tau + 1$, and generates the vehicle locations.

**Fig. 2 about here**

This research aims to develop an adjustment process to ensure that the traffic system moves to the desirable state in a stable manner while preserving the computational efficiency of the solution search process. To enable real-time deployment, the proposed methodology avoids the iterative solution framework by using the dynamical system in conjunction with time-dependent link performance functions. The travel costs are estimated based on the time-dependent link performance functions, updated at each $\tau$ to reflect the changes in traffic conditions. The virtual network states in the dynamical system can represent the current traffic conditions only or project the future conditions. Hence, this framework can be implemented in a reactive non-iterative model or a predictive non-iterative model depending on the information availability. Yang (2001) implements both models for a real traffic network and performs a comprehensive sensitivity analysis of their effectiveness for various parameters. Here, we focus on the reactive model, which uses the instantaneous travel costs.

2. **Stability and traffic assignment**

Smith (1979, 1984) first addressed the stability of traffic network equilibrium for the static traffic assignment problem. A dynamical system is used to model the route choice behavior. The Lyapunov function approach is used to study the stability of the equilibrium solutions. Horowitz (1984) proposed three models for the route choice decision-making process in a two-link network based on three weighted average measures. He defined the network equilibrium to be stable if the equilibrium point is unique and the convergence of link volumes to the equilibrium state from arbitrary initial points is guaranteed. Friesz et al. (1994) applied tatonnement adjustment processes from classical microeconomic equilibrium models to predict day-to-day changes in response to changes in demand. They analyzed the behavior of day-to-day trajectories from disequilibrium under complete or incomplete information provision using the tatonnement mechanism and discussed its stability properties. Zhang and Nagurney (1995, 1996; Nagurney and Zhang, 1996, 1997) introduced the projected dynamical system concept to study the route choice adjustment process in both elastic and fixed demand networks. The stationary point of such a dynamical system coincides with the user equilibrium flow pattern. They propose two distinct approaches: the monotonicity approach to analyze global stability and the regularity approach for local stability analysis. Watling (1999) extended Horowitz’s results (1984) to general networks. A dynamical adjustment process is proposed for studying the stability of the general asymmetric stochastic equilibrium assignment problem. The stability analysis concentrates on the linear approximation of the original nonlinear model, and can investigate only the local stability about the equilibrium point.

The literature vis-à-vis stability focuses primarily on static traffic assignment and the user equilibrium solution. In this paper, we propose time-dependent route guidance strategy models under the UE and SO assignment principles and address their stability using the Lyapunov approach. A route switching control strategy based on Smith (1984) is used to assign time-dependent traffic demand to the network. To the extent that SO or UE are desirable states under the corresponding objectives but probably rarely reached given the complexities inherent to dynamic traffic networks, our approach focuses on demonstrating that the proposed control strategies move the system toward the time-dependent stable desirable states prescribed by the objectives. The ability to address system stability over a set of desirable states rather than an isolated point, afforded by LaSalle’s theorem (Khalil, 1996), provides broader global insights while being synergistic in the context of real-time traffic operations for networks with advanced information systems. This is because the desirable state (for example, the instantaneous user equilibrium state) is a set of points along the time trajectory for dynamic networks.

An important contribution of the paper is that the Lyapunov functions $V(x)$, where $x$ represents the system state variable, for the SO and UE objectives are their corresponding objective functions under DTA. This overcomes the key difficulty of constructing a physically meaningful Lyapunov function for traffic systems akin to the use of different forms of energy to provide physical interpretations for mechanical systems. For example, the Lyapunov function $V(x)$ lacks intuitive physical meaning in Smith (1984) and Watling (1999), and hence is not straightforward to construct. In our analysis, we find that travel costs play an equivalent role in traffic assignment problems as energy does in mechanical systems. This affords a general stability analysis procedure for traffic assignment problems.

Another advantage of the proposed approach is its ability to determine less conservative estimates of the region of attraction. Smith’s (1984) Lyapunov function has a quadratic form which implies a region of attraction that satisfies $\{x \in R^n | V(x) \leq b\}$, where $b$ is a constant. Watling’s (1999) local stability analysis can imply a small region of attraction. In our study, the estimated region of attraction is a relevant compact invariant set, which is a less conservative estimation of the attraction region. Unlike most previous methods, our stability analysis requires only piecewise continuity in the dynamical system but not differentiability. Hence, the proposed procedure is
less restrictive vis-à-vis the inherently ill-behaved dynamic traffic networks.

In the context of real-time operations, the dynamical system search for the assignment strategy that leads to the stable desirable state (for the next assignment interval) can be truncated at any point in time. This is because the solution procedure always moves the system toward the time-dependent stable desirable state, implying that any intermediate solution is better than all previous intermediate solutions along the search trajectory. Therefore, the procedure is less computationally intensive than procedures that seek the equilibrium states. The ensuing trade-offs between computational time and solution effectiveness provide a convenient handle for practical implementation.

3. Problem description and methodology

Consider a dynamic traffic network \( G(N, A) \) where \( N \) is the set of nodes and \( A \) is the set of directed links. This study seeks to assign time-dependent paths to vehicles through a stable real-time information provision control strategy that satisfies some system-wide and/or individual user objectives. As discussed earlier, the assignment interval \( \tau \) denotes the duration for which the dynamical system is used to search for the stable path assignment strategy. Also, the dynamical system time unit \( t \) relates only to the solution procedure and has no mapping to the real system. Therefore, the number of vehicles in the real traffic network does not change along the \( t \) axis. The dynamical system searches for the stable desirable states using instantaneous travel costs. It should be noted here that the dynamical system analysis itself is generic and is not constrained by an instantaneous cost requirement. Thereby, it can be performed with the future predicted travel times or other externally specified costs (Yang, 2001).

Let \( \Omega \) be the set of O-D pairs in the network, and \( \omega \) denote an O-D pair, \( \omega \in \Omega \). The O-D pair from node \( n \) to node \( s, n, s \in A \), is denoted by \( \omega = (n, s) \). Let \( P \) be the set of paths in the network and \( P_{\omega} \) the set of paths for O-D pair \( \omega \), \( P_{\omega} \subset P \). Also, let \( p_{i,\omega} \) denote path \( i \) for O-D pair \( \omega \), \( p_{i,\omega} \in P_{\omega} \).

We now describe the dynamical system. The state variable \( x_{i,\omega}^\tau(t) \) in the dynamical system represents the flow at node \( n \) of path \( i \) for O-D pair \( \omega = (n, s) \) that is considered for assignment at time \( t \) in the dynamical system by the control strategy in assignment interval \( \tau \). To simplify notation, \( x_{i,\omega}^\tau = x_{i,\omega}^\tau(t) \) in the dynamical system because the analysis is performed for interval \( t \) in the dynamical system. It includes two components: (1) new demand originating at node \( n \) initially assigned to path \( i \) for the assignment interval \( \tau \), and (2) existing flow to destination \( s \) that reaches node \( n \) of path \( i \) in interval \( \tau \). Thereby, \( n \) is the origin node for the new demand and the switching decision node for the existing flow. This implies that vehicles can be re-routed en-route at decision nodes. A key implication of this is that the experienced path of any vehicle may be generated through a combination of instantaneous and experienced link costs. Hence, the use of instantaneous link costs in the solution procedure does not necessarily imply that the experienced path is the initially determined instantaneous cost path. For the new demand that seeks to enter the network in assignment interval \( \tau \), path \( i \) is a preliminary externally determined initial path (for example, based on historical traffic flow data) that may be re-assigned based on the instantaneous traffic conditions in interval \( \tau \). In the absence of historical data, these initial paths could be determined using the path assignment proportions specified by the control strategy in interval \( (\tau - 1) \). Let \( x_{i,\omega}^\tau(t) \) represent the vector of flows \( x_{i,\omega}^\tau(t) \) on all paths \( i, \ i \in P_{\omega} \), for O-D pair \( \omega \) at time \( t \) in the dynamical system; \( x_{i,\omega}^\tau = x_{i,\omega}^\tau(t) \). Then, \( d_{\omega}^\tau = \sum_{i \in P_{\omega}} x_{i,\omega}^\tau \) represents the total traffic flow for O-D pair \( \omega \) that is considered for assignment by the control strategy in assignment interval \( \tau \).

By definition, \( d_{\omega}^\tau \) is the instantaneous demand for the interval \( \tau \) and is constant in the virtual system along the \( t \) axis. The vector of \( d_{\omega}^\tau \) for all O-D pairs \( \omega, \ \omega \in \Omega \), is denoted by \( d^\tau \). Also, let \( x^\tau(t) \) denote the vector of flows \( x_{i,\omega}^\tau(t) \) for all O-D pairs \( \omega \in \Omega \) at time \( t \) in the dynamical system, \( x^\tau(t) = x^\tau(t) \). Hence, \( x^\tau \) represent the flows available for assignment in interval \( \tau \).

Let \( f^\tau_a(t) \) denote the traffic flow contribution on link \( a \) due to path flows \( x^\tau(t) \) at time \( t \) in the dynamical system; \( f^\tau_a = f^\tau_a(t) \), and \( f^\tau \) represent the vector of all link flows \( f^\tau_a \), \( \forall a \in A \), at time \( t \) in the dynamical system. Hence, \( f^\tau(t) \) represents the vector of link flows at time \( t \) in the dynamical system due to the path flows \( x^\tau \) being assigned.

As discussed earlier, the total number of vehicles is fixed in the dynamical system associated with an assignment interval \( \tau \). Hence, the virtual network in the dynamical system along the \( t \) axis can be treated as a static traffic network. Thereby, static link-path incidence relationships and flow conservation equations can be applied to it. The link-path incidence variables are represented by \( \delta_{i,a}^\tau \). They take a value 1 if link \( a \) is on path \( i \) of O-D pair \( \omega \), and 0 otherwise. The associated link-path incidence matrix is denoted by \( \delta \). The relationship between \( f^\tau(t) \), \( \delta \), and \( x^\tau \) is given by:

\[
\dot{x}^\tau = \delta x^\tau. 
\]

For a particular \( f^\tau_a \), we have

\[
f^\tau_a(t) = \sum_{\omega \in \Omega} \sum_{i \in P_{\omega}} \delta_{i,a}^\tau x^\tau(t). 
\]

Let \( c_u(t) \) represent the instantaneous link performance function on link \( u \) in assignment interval \( \tau \), and \( c_u^\tau \) represent the instantaneous travel cost on link \( a \). A link performance function describes the relationship between link flows and travel costs. The typical definition of the link performance function \( c_u(t) \) assumes it to be a function of just the flow on link \( a \). Here, we consider link interactions, of essence to modeling dynamic networks, and assume \( c_u(t) \) to be a function of all link flows \( f^\tau \).

Since traffic conditions vary with time in dynamic traffic networks, link performance functions are time-dependent. The instantaneous link travel cost \( c_u^\tau \) is due to two
components: (1) the previously assigned flow that is still present in the network and is not considered for assignment or has no opportunity to switch paths in assignment interval $\tau$, labeled background flow $b^*_a(\tau)$; and (2) the flow $x^i(\tau)$ seeking assignment in the interval $\tau$. Therefore, in the context of the dynamical system, $b^*_a(\tau)$ is a fixed quantity along the $t$ axis for a given $\tau$ as the associated vehicles cannot switch routes in $\tau$. Since $b^*_a(\tau)$ is time-invariant in the dynamical system in the interval $\tau$, it can be denoted simply as $b^*_a$. Hence, $b^*_a$ is the background flow on link $a$ in assignment interval $\tau$ in the dynamical system and $b^*$ represents the vector of background flows $b^*_a$. As illustrated in Fig. 3, they are necessary to correctly determine the link travel times for assigning the O-D flows in each $t$ of the dynamical system corresponding to $\tau$, but are not decision variables in terms of $t$. Let $f^j_a(\tau)$ denote the total traffic flow on link $a$ in assignment interval $\tau$ at time $t$ in the dynamical system, and $\vec{f}^j(\tau)$ the vector of all $f^j_a(\tau)$. The relationship between $f^j_a(\tau)$, $b^*_a$, and $\vec{f}^j(\tau)$ is given by:

$$\vec{f}^j(\tau) = f^j_a(\tau) + b^*_a.$$  (3)

Fig. 3 about here

$f^j_a(\tau)$ depends on the route guidance strategy for the dynamical system interval $\tau$. Since $b^*_a$ is fixed in $\tau$ and the relationship between $f^j_a(\tau)$, $b^*_a$, and $\vec{f}^j(\tau)$ is known, $c^*_a(\tau)$ can be transformed to a function of $f^j_a(\tau)$. This transformation simplifies the theoretical analysis in later sections. Fig. 3 illustrates the transformation of $c^*_a(\tau)$, obtained by shifting the $y$-axis to $b^*_a$ for each $f^j_a(\tau)$. Therefore, $c^*_a(\tau)$ can be represented as a function of $f^j_a(\tau)$, denoted as $c^*_a(f^j_a(\tau))$, and $c^*(f^j_a(\tau))$ is the vector of the projected instantaneous link travel costs on all links. $c^*_a(\tau)$ is assumed here to be positive, continuously differentiable, and monotonically increasing with respect to each $f^j_a$, for all $a \in A$. It explicitly considers links interactions but does not make limiting assumptions on them, precluding the guarantee of uniqueness for the equilibrium state as can be expected in general traffic networks. $c^*_a(\tau)$ is asymptotic to the link capacity implying that as the traffic flow approaches capacity, a very high penalty is incurred. The shape of the link performance function is assumed to be fixed for an assignment interval. Since the capacity and the ambient traffic conditions may vary over time due to complexities inherent to dynamic traffic networks the shape of the link performance function may change from one assignment interval to the next. However, it is always monotonically increasing and asymptotically approaches capacity. The monotonic increasing and continuous differentiability assumptions do not necessarily hold across assignment time intervals. This implies that for the same link flow the link travel costs may be different in different assignment intervals, which is consistent with traffic dynamics in time-dependent networks. These characteristics of link performance functions are highlighted in Fig. 4.

From the perspective of real-time traffic operations, the above modeling approach for $c^*_a(\tau)$ has several advantages vis-à-vis adequately replicating traffic flow dynamics. The use of time-dependent link performance functions circumvents the key difficulty of violating the monotonic increasing and/or continuous differentiability assumptions across time intervals while enabling an adequate representation of ambient traffic conditions. This is especially critical under highly congested situations where traffic conditions in the vicinity of a link can significantly influence its travel costs, implying significant link interactions. A synergistic modeling aspect in this regard is the assumption that the travel time on a link $a$ depends on the flows on all links. The use of time-dependent link performance functions for DTA has previously proved to be an effective and adequate approach (Peeta, 1994; Peeta and Mahmassani, 1995b) for projecting time-dependent conditions. The potential limitation of assuming a monotonically increasing and continuously differentiable link performance function that is constant within an assignment interval is the possible lack of robustness in accounting for significant traffic events (such as incidents) which occur during that assignment interval. This suggests smaller assignment intervals to neutralize this issue, leading to the usual trade-offs between computation and accuracy vis-à-vis implementation. Yang (2001) analyzes assignment intervals varying from 3 to 15 minutes for a specific network; they highlight the trade-offs between solution effectiveness and computational time.

Let $C^{\tau}(x^i(\tau))$ represent the path travel cost for traffic flow $x^i(\tau)$ at time $t$ in the dynamical system; $C^\tau = \sum_{a \in P} c^*_a(\tau)$. Since instantaneous link travel costs are used, the path travel costs are also instantaneous. $C^{\tau}(x^i(\tau))$ represents the vector of path travel costs for all paths of the O-D pair $\omega$ at time $t$ in the dynamical system corresponding to the traffic flow $x^i(\tau)$; $C^{\tau} = \sum_{a \in P} c^*_a(\tau)$, and $C^\tau(x^i(\tau))$ represents the vector of path travel costs $C^{\tau}(x^i(\tau))$ for all O-D pairs at time $t$ in the dynamical system; $C^\tau = \sum_{a \in P} c^*_a(\tau)$. Based on the link-path incidence relationship, we have

$$C^\tau = \delta^T \cdot C^\tau.$$  (4)

Let $\dot{x}^{\tau}(\tau)$ represent the switch rate of the traffic flow $x^{\tau}(\tau)$ from path $f$ to path $i$ at time $t$ in the dynamical system; $\dot{x}^{\tau}_{ij}(\tau) = \dot{x}^{\tau}_{ij}(\tau)$. $\dot{x}^{\tau}(\tau)$ is the net switch rate for path $i$ of O-D pair $\omega$ at time $t$ in the dynamical system; $\dot{x}^{\tau}_{ij}(\tau) = \dot{x}^{\tau}_{ji}(\tau)$.

4. Basic route switching strategy

In this section, we first state the assumptions behind the basic route switching control strategy and then construct the model for a pair of paths. The existence of solutions is discussed, followed by the extension to general networks.
4.1 Basic route switching model

The route guidance control models we propose are based on the following assumption: the switch rate of traffic from path \( j \) to path \( i \) at node \( n \) at time \( t \) in the dynamical system is proportional to the path cost difference, and the traffic flow \( x_{ij}^{to} (t) \) being considered for assignment. However, if the current route has the lesser cost, the drivers do not switch. In this context, every decision node \( n \) and destination \( s \) can be treated as an O-D pair, i.e. \( \omega = (n, s) \in \Omega \). Mathematically, the route switching principle can be expressed as:

\[
\dot{x}_{ij}^{to} (t) = \left\{ \begin{array}{ll}
\alpha (C_{ij}^{to} - C_{ij}^{noc}) x_{ij}^{to} (t), & \text{if } C_{ij}^{to} - C_{ij}^{noc} > 0 \\
0, & \text{if } C_{ij}^{to} - C_{ij}^{noc} \leq 0 ,
\end{array} \right.
\]

\( \alpha > 0. \)  \( (5) \)

We introduce a function \( \phi(y_1, y_2) \) which is defined as:

\[
\phi(y_1, y_2) = \begin{cases} 
 y_1 - y_2 , & \text{if } y_1 - y_2 > 0 \\
0 , & \text{if } y_1 - y_2 \leq 0.
\end{cases}
\]

Then, the basic route switching model can be re-stated:

\[
\dot{x}_{ij}^{to} (t) = \alpha \phi(C_{ij}^{to}, C_{ij}^{noc}) x_{ij}^{to} (t).
\]  \( (7) \)

Using different values for \( \alpha \) and applying different measures of the cost \( C_{ij}^{noc} \), we can formulate different route switching control strategies. Since the right-hand side of (7) is continuous, there exists a solution \( x_{ij}^{to} (t) \) to the ordinary different equation (7) as a well-defined dynamical system (Kahlil, 1996).

4.2 Route switching model for general networks

Model (7) describes the magnitude and direction of the flow switching between two paths. Based on equation (7), for path \( i \), the switching rate from other paths to path \( i \) would be positive and the switching rate from path \( i \) to other paths would be negative. The net switch rate for path \( i \) at decision node \( n \) at time \( t \) can be obtained by summing up all positive switching rates and negative switching rates:

\[
\dot{x}_{i}^{to} (t) = \sum_{j \in \mathcal{E}_{i}} \alpha \phi(C_{ij}^{to}, C_{ij}^{noc}) x_{ij}^{to} (t) - \sum_{j \in \mathcal{E}_{i} \setminus \{i\} \cup \mathcal{D}_{i}} \alpha \phi(C_{ij}^{to}, C_{ij}^{noc}) x_{ij}^{to} (t).
\]  \( (8) \)

The first term in equation (8) is the summation of all switch-in rates for path \( i \) and the second term is the summation of the switch-out rates.

Furthermore, we can consider the general network with multiple O-D pairs. By lumping the route switching model (8) for each O-D pair, the route switching model for a general network can be expressed as:

\[
\dot{x}(t) = \Phi^s(x(t)) \cdot x(t),
\]  \( 9a \)

\[
\Phi^s(x) = \begin{bmatrix}
\Phi^{r1}(x) & 0 & \cdots & 0 \\
0 & \Phi^{r2}(x) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Phi^{rD}(x)
\end{bmatrix},
\]

and

\[
\dot{x}_{i}^{to} (t) = \begin{bmatrix}
- \sum_{p \in \mathcal{D}_{i}} \alpha \phi(C_{ip}^{to}, C_{ip}^{noc}) \\
\alpha \phi(C_{ip}^{to}, C_{ip}^{noc}) - \sum_{q \in \mathcal{D}_{i} \setminus \{p\}} \alpha \phi(C_{pq}^{to}, C_{pq}^{noc}) \\
\cdots \\
\cdots
\end{bmatrix} \cdot \begin{bmatrix}
x_{i}^{to} (t)
\end{bmatrix}.
\]  \( 9c \)

\( \dot{x}_{i}^{to} \) is continuous because it is a summation of \( \dot{x}_{ij}^{to} \) (see system 9) which are continuous. Therefore, each element \( \dot{x}_{i}^{to} \) in system (9) is continuous. Consequently, the solution existence for system (9) is guaranteed.

5. Route guidance control strategy models

In this section, two different route guidance control strategies are investigated based on system (9). The user equilibrium objective model uses the instantaneous UE states as desired network states and aims to push the system towards these states. The system optimal objective model drives the system to the instantaneous SO states.

5.1 User equilibrium objective model

In a traffic network, user equilibrium (Wardrop, 1952) is reached when no driver can improve his/her travel time by unilaterally switching routes. It is characterized by travel times for used paths being equal, and less than or equal to the travel times on unused paths. The UE goal can be addressed by using the instantaneous path travel times \( T_i \) to represent the travel costs \( C_i \) in model (9). We denote \( T_i \) as \( T_i \). The basic route switching principle for the UE objective model is:

\[
\dot{x}_{ij}^{to} (t) = \dot{\phi}(T_i, T_i^{to}) x_{ij}^{to} (t), \quad \text{for all } i \neq j.
\]  \( (10) \)

By replacing \( C_i^{to} \) by \( T_i^{to} \) in model (9), the user equilibrium objective model can be formulated as:

\[
\dot{x}(t) = \dot{\Phi}^u(x) \cdot x(t).
\]  \( (11) \)

5.2 System optimal objective model

Following Wardrop’s (1952) second principle, the used paths and the associated flows under system optimality are such that the total system travel time is minimized. The system optimal state is characterized by used paths for an O-D pair having the same marginal travel costs, and the unused paths having equal or higher marginal travel costs.

The instantaneous marginal travel time \( \dot{T}_i^{to} \) is defined as the contribution of an additional user on path \( i \) to the total system travel time under current traffic conditions. A comprehensive discussion on marginal travel times is provided in Peeta (1994). The instantaneous marginal travel time can be expressed as:

\[
\dot{T}_i^{to} = \frac{\partial}{\partial \lambda} \left( \sum_{u \in \mathcal{N}} f^u_i(t) \cdot e^u_i(f^u_i(t)) \right).
\]  \( (12) \)

If the travel costs in the basic route switching model (7) are expressed in terms of the instantaneous marginal travel times, the system moves towards the dynamic
instantaneous SO state under the associated control strategy. The SO objective model can be formulated by replacing $T_{i}^{\text{fso}}$ by $\bar{T}_{i}^{\text{fso}}$ in (11).

$$\dot{x}(t) = \Phi_{\text{so}}(x) \cdot x(t).$$  \hspace{1cm} (13)

5.3 Significance of the equilibrium state

In this section, we show that the equilibrium state of system (11) corresponds to the instantaneous UE solution and that of system (13) is equivalent to the instantaneous SO solution. We also discuss another traffic condition that leads the system to $\dot{x}(t) = 0$.

**Definition 1.** (Equilibrium State) Consider a dynamical system $\dot{x}(t) = F(x(t))$, where $x(t)$ represents the state variable. An equilibrium state of the dynamical system is a constant for all time, and satisfies $x(t) = \chi^c$, $t \in R$. Since the equilibrium solution must satisfy this differential equation, $\dot{x}(t) = F(x(t))$, all equilibrium states are given by $\dot{x}(t) = F(x^c) = 0$ (Khalil, 1996).

Let us suppose that at least some users have the opportunity to switch their routes. According to **Definition 1**, the equilibrium state for system (9) is:

$$\dot{x}(t) = \Phi^{\text{fso}}(x) \cdot x(t),$$  \hspace{1cm} (14)

where $x(t)$ is the equilibrium state. Suppose path $k$ has the least travel cost for O-D pair $\omega$. Then, according to model (8), no vehicle would switch from path $k$. It implies that

$$\sum_{i \in P_{\omega} \setminus \{k\}} \alpha \phi(C_{i}^{\text{fso}}, C_{k}^{\text{fso}}) x_{k}^{\omega} = 0.$$  \hspace{1cm} (15)

Since path $k$ is the best route:

$$C_{j}^{\text{fso}} \geq C_{k}^{\text{fso}} \geq 0 \text{ and } x_{j}^{\omega} \geq 0, \ \forall j \in P_{\omega} \setminus \{k\}. \hspace{1cm} (16)$$

In order to let $\dot{x}_{k}(t) = 0$ in equation (15), it must follow that

$$C_{j}^{\text{fso}} \begin{cases} = C_{k}^{\text{fso}}, & \text{if } x_{j}^{\omega} > 0 \\ \geq C_{k}^{\text{fso}}, & \text{if } x_{j}^{\omega} = 0. \end{cases} \hspace{1cm} (17)$$

If the travel cost is represented by the instantaneous path travel time $T_{i}^{\text{fso}}$, the conditions (17) imply user equilibrium. If the travel cost is represented by the instantaneous path marginal travel time $\bar{T}_{i}^{\text{fso}}$, conditions (17) denote system optimality. Hence, the equilibrium states of the dynamical systems (11) and (13) correspond to the instantaneous UE and SO solutions, respectively.

When the network gridlocks, users do not have opportunities to switch their routes. Then, the route switching rate for each path is forced to be zero because of the gridlock, that is, $\dot{x}(t) = 0$. Such a scenario would require interventionist control strategies beyond route switching to alleviate the situation.

Hence, $\dot{x}(t) = 0$ implies that either the system reaches the equilibrium state if opportunities to switch exist, or it is gridlocked.

6. System properties

We discuss two important properties of route guidance control models: solution non-negativity and flow conservation. The traffic flows along paths are always non-negative. In other words, only solutions in the set $X^+ = \{x^c \in R^n | x^c_i \geq 0, \forall i \in P, \omega \in \Omega\}$ have a physical meaning. We show that if the initial point originates in the set $X^+$, all solutions prescribed by system (9) remain inside it.

**Property 1.** (Solution non-negativity) If system (9) initiates at a non-negative state, all solutions are non-negative. Mathematically it can be stated as,

$$x^c(0) \in X^+ \Rightarrow x^c(t) \in X^+, \ \forall t \geq 0,$$  \hspace{1cm} (16)

where

$$X^+ = \{x^c \in R^n | x^c_i \geq 0, \forall i \in P, \omega \in \Omega\}.$$  \hspace{1cm} (17)

**Proof.** The boundary points of $X^+$ are $x^c_j(t) = 0$ and $x^c_j(t) \geq 0$ for all $j \neq i$. Let us analyze equation (8). At the boundary point, $x^c_i(t) = 0$; so

$$\sum_{j \in P_{\omega} \setminus \{i\}} \alpha \phi(C_{i}^{\text{fso}}, C_{j}^{\text{fso}}) x_{j}^{\omega} = 0.$$  \hspace{1cm} (18)

Therefore, $x^c_j(t) = 0$ at the boundary points. Hence, for every $x^c_j(t) = 0$, $x^c_i(t) \geq 0$.

Since $x^c_j(t) = 0$ at boundary points, it implies that when the system (9) reaches a boundary point of $X^+$, $x^c_i(t)$ is forced to increase its value or stay at the boundary point. Hence, if the initial state $x^c(0) \in X^+$, all solutions $x^c$, $\forall t \geq 0$, remain in $X^+$. This implies that non-negativity is satisfied in our system.

Next, we show that the solutions of system (9) always reside in $M = \{x^c \in R^n | x^c_j \in X^+ \text{ and } \sum_{i \in P_{\omega}} x^c_i = d^\text{fso} \}$. In other words, the subset $M$ is an invariant set of system (9).

A set $M$ is said to be an invariant set of a system if an initial solution inside $M$ implies that all solutions will reside in $M$. It implies that for each O-D pair, all solutions of system (9) satisfy the O-D flow conservation constraint. As discussed earlier, in our dynamical system, O-D flow does not refer to just the external new demand entering the traffic system. Rather, it is the flow being considered for assignment at the start node of each O-D pair in assignment interval $\tau$.

**Property 2.** (Flow conservation) The subset

$$M = \{x^c \in R^n | x^c_j \in X^+ \text{ and } \sum_{i \in P_{\omega}} x^c_i = d^\text{fso} \}$$  \hspace{1cm} (19)

in system (9)
is an invariant set. That is, \( x^r(0) \in M \Rightarrow x^r(t) \in M \), \( \forall t \geq 0 \).

**Proof.** If \( x^r \in X^+ \), system (9) states that:
\[
\sum_{i \in P_a} x_i^{T_0} = -\sum_{i \in P_a, \{j\}} \alpha \phi(C_{i,j}^{T_0}, C_i^{T_0}) x_i^{T_0} + \sum_{i \in P_a, \{j\}} \alpha \phi(C_{i,j}^{T_0}, C_i^{T_0}) x_j^{T_0} - \sum_{i \in P_a, \{j\}} \alpha \phi(C_p^{T_0}, C_i^{T_0}) x_p^{T_0} + \sum_{i \in P_a, \{j\}} \alpha \phi(C_p^{T_0}, C_i^{T_0}) x_j^{T_0} \]
\[
= 0.
\]
(18)
Since \( \sum_{i \in P_a} \dot{x}_i^{T_0}(t) = 0 \), it follows that \( \sum_{i \in P_a} x_i^{T_0}(t) \) is a constant. Hence, if \( \sum_{i \in P_a} \dot{x}_i^{T_0}(0) = d^{T_0} \), then
\[
\sum_{i \in P_a} x_i^{T_0}(t) = d^{T_0}, \ \forall t \geq 0.
\]
Also, from Property 1, we know \( x^r(0) \in X^+ \Rightarrow x^r(t) \in X^+ \), \( \forall t \geq 0 \). Hence,
\[
x^r(0) \in M \Rightarrow x^r(t) \in M, \ \forall t \geq 0; \ M \) is an invariant set.

By definition, the function \( \phi(y_1, y_2) \) in equation (6) may not be differentiable at \( y_1 - y_2 = 0 \). Hence, systems (9), (11), and (13) may be non-differentiable. A non-differentiable system cannot be linearized about the equilibrium points to study the local stability characteristics about the equilibrium states. Therefore, we apply the classical technique from non-linear dynamical systems, the Lyapunov function approach, to analyze our route guidance control models.

7. **Stability analysis**

LaSalle’s Theorem is used to address the stability properties of route guidance models. It can be stated as the follows.

**Theorem 1.** (LaSalle’s Theorem) Consider a dynamical system \( \dot{x}(t) = F(x) \). Let \( V(x): \mathbb{R}^n \rightarrow \mathbb{R} \) be a continuously differentiable, radially unbounded function such that \( V(x) \leq 0 \) for all \( x \). A scalar valued function \( V(x) \) is said to be radially unbounded if \( \lim_{x \rightarrow \infty} V(x) = \infty \). Let \( S = \{ x \in \mathbb{R}^n | \dot{V}(x) = 0 \} \), and \( E \) be the largest invariant set contained in \( S \). Then all solutions of system \( \dot{x}(t) = F(x) \) are bounded and converge to \( E \) (Khalil, 1996).

The function \( V(x) \) is called the Lyapunov function, \( \dot{V}(x) \) is the derivative of \( V(x) \) along the trajectories of the system \( \dot{x}(t) = F(x) \), defined as \( \dot{V}(x) = \frac{dV}{dt}(x) \). The challenge for the stability analysis is to construct meaningful Lyapunov functions. In general, the Lyapunov function relates to some form of energy in classical mechanical systems. Previous studies (Watling, 1999) have stated the difficulty in constructing physically meaningful Lyapunov functions in the context of traffic assignment. Studies that used a specific Lyapunov function do not attribute physical significance to it. Our study finds that the objective function associated with an assignment objective in a traffic network plays a similar role as energy does in mechanical systems. The Lyapunov functions for both the UE and SO objective models are inspired by the time-dependent objective functions of the corresponding DTA models (Peeta and Mahmassani, 1995b).

7.1 **Stability property of the UE objective model**

The candidate Lyapunov function that we propose for the user equilibrium objective model is the following line integral function:
\[
V(x(t)) = \int_{\theta_0}^{T(\eta)} c^r(\eta)^T d\eta,
\]
where \( \eta \) is a vector of dummy variables representing link flows and is differentiated from \( f^r(t) \) in (19) which represent the actual values of link traffic flows, and \( c^r(\eta)^T d\eta \) denotes the inner product of vector \( c(\eta)^T \) and vector \( d\eta \). Equation (19) is a line integral of vector \( c^r(\eta) \) on the curve of 0 to \( f^r(t) \). It corresponds to the standard UE assignment objective function. Based on equation (2), \( f^r(t) \) is a linear transformation of \( x^r(t) \); therefore, as \( \| f^r(t) \| \rightarrow \infty \), \( \| f^r(t) \| \rightarrow \infty \). Also, \( c^r(\eta)^T d\eta \) is a monotonically increasing function with respect to each \( f_a^r(t) \), \( \forall a \in A \). Hence, \( \int_{\theta_0}^{T(\eta)} c^r(\eta)^T d\eta \rightarrow \infty \) as \( \| f^r(t) \| \rightarrow \infty \). Thus, \( \lim_{t \rightarrow \infty} V(x^r(t)) = \infty \). \( V(x^r(t)) \) is a radially unbounded function. Equation (19) can be expanded as:
\[
V(x^r(t)) = \int_{\theta_0}^{T(\eta)} c^r(\eta)^T d\eta = \int_{\theta_0}^{s} c^r(\eta)^T d\eta dt
\]
\[
= \int_{\theta_0}^{s} c^r(\eta)^T \frac{d\eta}{ds} ds.
\]
(20)
Here, \( t_0 \) is the initial time in the dynamical system corresponding to the initial state \( x^r(0) \). \( t^r \) is the time in dynamical system when the system reaches the state \( x^r(t) \); the corresponding link flows are \( f^r(t) \).

Equation (20) converts (19) to an integral along a path in the plane parameterized by \( t \). The conversion is to avoid dealing with the complex interactions among \( f_a^r \). Hence, the derivative of \( V(x(t)) \) along the trajectories of the system is:
\[
\dot{V}(x^r(t)) = \frac{dV}{dt}(x^r(t)) = \frac{d}{dt}\left[\int_{\theta_0}^{s} c^r(\eta)^T \frac{d\eta}{ds} ds\right].
\]
(21)
The second term of the integrand in equation (21) is
\[
\frac{dx^T}{dt} = \left( T^T \cdot \dot{x}^T \right)\ .
\]
Therefore, the first two terms of the integrand in equation (21) become:
\[
(c^T (f^T \delta)) = (c^T (f^T \delta)) = (c^T (f^T \delta)) = (C^T)\ .
\]
In the UE model, the instantaneous travel time \( T^{ins}_i \) is used to represent the travel cost \( c_{ij}^{ins} \). The third term of the integrand in equation (21) is
\[
\frac{dx^T}{dt} = \dot{x}^T \cdot \dot{x}^T \ .
\]
Substituting into equation (23):
\[
\dot{V}(x^T(t)) = (T^T) \cdot \Phi_{UE} \cdot x^T(t)\ .
\]
For each O-D pair \( \omega \), expanding \( (T^T) \cdot \Phi_{UE} \cdot x^T \):
\[
\dot{V}(x^T(t)) = \sum_{i=1}^{\Omega} \sum_{j=1}^{\omega} \alpha(T_{i,j}^T - T_{i,j}^T) \Phi(T_{i,j}^T, T_{i,j}^T) \cdot x_{i,j}^T \ .
\]
There are three possibilities for the relationship between \( T_{i,j}^T \) and \( T_{i,j}^T \) in equation (25):
\[
T_{i,j}^T - T_{i,j}^T > 0 \quad \text{and} \quad x_{i,j}^T \geq 0, \quad x_{i,j}^T \geq 0, \quad \alpha > 0, \quad \forall \ i \neq j, \omega, \text{then}
\]
\[
\alpha(T_{i,j}^T - T_{i,j}^T) \Phi(T_{i,j}^T, T_{i,j}^T) \cdot x_{i,j}^T = 0 \ .
\]
Thus,
\[
\alpha(T_{i,j}^T - T_{i,j}^T) \Phi(T_{i,j}^T, T_{i,j}^T) \cdot x_{i,j}^T \leq 0 \ .
\]
Equation (28) is equivalent to the instantaneous UE conditions. Hence, \( S = \{ x^T \in \mathbb{R}^n \mid V(x^T(t)) = 0 \} \) corresponds to the set of instantaneous UE states.

From Property 2, \( M \) is the invariant set with respect to the O-D pair in system (11). Since \( S \) is the set of instantaneous UE states, it follows that \( M \cap S = S \). The largest invariant set in \( S \) is \( S \) itself. Therefore, by Theorem 1, all solutions of the dynamical system (11) are bounded and converge to \( S \) which contains the instantaneous UE solutions for different \( d^T \). Each element in \( S \), for a given \( d^T \), corresponds to the instantaneous UE state in terms of the invariant set \( M \). Hence, all solutions of the dynamical system (11) converge to their corresponding instantaneous UE states. The UE objective model (11) is stable about the manifold of instantaneous UE states defined by the corresponding time-dependent demand.

7.2 Stability property of the SO objective model

The candidate Lyapunov function that we propose for the system optimal objective model is:
\[
V(x^T(t)) = \sum_{a=1}^{A} f_{a}^T (c_{a}^T (f^T \delta)) \ .
\]
It represents the total system travel time. Since \( c_{a}^T (f^T \delta) \) is a monotonically increasing function of \( f_{a}^T (t) \ \forall a \in A \) and \( f^T (t) \) is a linear transformation of \( x^T (t) \), \( V(x^T(t)) \rightarrow \infty \) as \( \|x^T(t)\| \rightarrow \infty \). Hence, \( V(x^T(t)) \) in equation (29) is a radially unbounded function. The derivative of \( V(x^T(t)) \) along the trajectories of the system is:

\[
\dot{V}(x^T(t)) = \frac{dV}{dt}(x^T(t)) = \frac{dx^T}{dt} \cdot \dot{x}^T(t) = DV(x^T(t)) \cdot x^T(t) \ .
\]

We first compute \( DV(x^T(t)) \) in equation (30). From equation (12), we know that
\[
\frac{\partial V}{\partial x_{i,j}^T}(x^T) = \left( \sum_{a=1}^{A} f_{a}^T (c_{a}^T (f^T \delta)) \right) \ .
\]

Following the logic used in the stability analysis of the user equilibrium objective model,
\[
(T^{ins}_i)^T \Phi_{SO}(x^T) = \sum_{i=1}^{\Omega} \sum_{j=1}^{\omega} \alpha(T_{i,j}^T - T_{i,j}^T) \Phi(T_{i,j}^T, T_{i,j}^T) x_{i,j}^T \ .
\]
\[
\dot{V}(x^r(t)) = (\hat{T}^r)^T \cdot \Phi_{SO}^r \cdot x^r(t)
\]
\[
= \sum_{a \in \Omega} \sum_{i \in \partial a} \sum_{j \in \partial a} \alpha(\hat{T}^r_{ij} - \check{T}^r_{ij}) \phi(\hat{T}^r_{ij}, \check{T}^r_{ij}) x^r_{ij} \cdot \eta^r_{ij}. \tag{34}
\]

Using the same reasoning used in the stability analysis of the user equilibrium objective model, we can show
\[
\dot{V}(x^r(t)) = \sum_{a \in \Omega} \sum_{i \in \partial a} \sum_{j \in \partial a} \alpha(\hat{T}^r_{ij} - \check{T}^r_{ij}) \phi(\hat{T}^r_{ij}, \check{T}^r_{ij}) x^r_{ij} \eta^r_{ij} \leq 0. \tag{35}
\]

And the set \( S = \{ x^r \in \mathbb{R}^n \mid \dot{V}(x^r(t)) = 0 \} \) corresponds to the set of instantaneous SO solutions for different \( d^r \).

Therefore, all solutions of dynamical system (13) are bounded and converge to \( S \). Hence, the SO objective model (13) is stable about the manifold of instantaneous SO solutions defined by the corresponding time-dependent demand.

Based on our link performance function assumptions and the proposed route guidance models, solution uniqueness cannot be guaranteed. Hence, multiple equilibria can exist based on our models, as can be expected in general traffic networks. Since LaSalle’s Theorem can address a system which is stable about a set of equilibrium states rather than an isolated equilibrium point, our stability analysis procedure has the ability to address traffic networks. Since LaSalle’s Theorem can address a system toward the prescribed objective based on the current network conditions. LaSalle’s theorem, an extension of the classical Lyapunov approach, is used to perform the stability analysis. The analysis addresses the global behavior of the route guidance control strategies.

Of significance to real-time traffic operations, the approach ensures stability by moving the system towards the equilibrium states prescribed by the corresponding objectives rather than seeking the equilibrium states themselves. This is a realistic operational perspective because the equilibrium states may never be reached in practice given the complexities inherent to dynamic traffic networks. This implies three advantages in the context of real-time operations. First, computational efficiency is ensured by circumventing the determination of the equilibrium states themselves. Second, by focusing on a set of equilibrium states defined by different demands, it enables addressing problems with time-varying demand. Third, by addressing a set of equilibrium states, it lends a global perspective to the stability analysis unlike approaches that are restricted to the vicinity of a specific equilibrium state. A significant contribution of the paper is the physical interpretation of the Lyapunov function in the context of dynamic traffic assignment problems. This provides a general framework for stability analysis of traffic assignment problems.

Yang (2001) conducted a detailed analysis of the performance of the proposed stable reactive route switching control model and two benchmark rolling horizon models that include or exclude en-route re-routing. The results suggest that the system performance under the proposed stable control model for dynamic traffic assignment is comparable to that under the rolling horizon models though it does not project future traffic conditions. Of critical significance to real-time tractability, its computational times are less by factors of 5-8 compared to the rolling horizon models. Yang (2001) extends the reactive model to include a predictive capability of the near-term future traffic conditions, leading to an internal model control based framework.

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References


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Assignment interval $r$