Theoretical Aspects of a Stable Dynamic Traffic Assignment Strategy

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Abstract

This paper addresses the theoretical aspects of a dynamical systems based stable dynamic traffic assignment (DTA) strategy for traffic networks equipped with advanced information systems. A non-linear dynamical system formulation is proposed. The solution existence is analyzed. The concept of time-dependent link performance functions is used to capture the time dependency of dynamic traffic networks. The assumptions on the link performance functions and their implications for the stable DTA strategy are discussed. Insights on the model behavior and system properties are illustrated using a two-link network. The Lyapunov-based theoretical stability analysis is related to the conceptual stability definitions.

Introduction

This study addresses the DTA problem faced by a central controller in a traffic network system who has information in real-time on current traffic conditions and time-dependent origin-destination (O-D) desires, and is seeking to provide real-time routing information to suitably equipped users while satisfying some system-wide and/or individual user objectives. Since information provision represents an external control variable that influences system performance, the stability of the system under the proposed route guidance control strategy is an important operational issue. Previous efforts to address the DTA problem focus on determining the desired final solution without necessarily addressing how the traffic system moves to it. Given the transient nature of peak traffic periods, the traffic system could deviate from the desired final solution as time progresses. This study not only addresses the solution that satisfies the prescribed objective, but also aims to ensure that the traffic system approaches the solution in a stable manner. In this context, the notion of stability implies that: (1) all solutions are bounded and converge to the time-dependent desirable states which are the solutions of the route guidance control strategy model at different instants of time; (2) the proposed control strategies minimize or limit the deterioration of system performance; and (3) if perturbations are introduced to the traffic network, the system reaches an equilibrium after
the perturbations are removed. While a guarantee of stability is unrealistic for operational traffic networks given the various factors that influence it, this study seeks insights on the effects of information provision. We show that the proposed route guidance control strategy moves the system towards the desirable time-dependent stable states, should such states exist.

Smith [17, 18] first addressed the stability of traffic equilibria for the static traffic assignment problem. He used a dynamical system to model route choice behavior and the Lyapunov function approach to study the stability of the user equilibrium (UE) solution. Horowitz [5] proposed three models for the route choice decision-making process in two-link networks based on three weighted average measures. He defined the equilibrium to be stable if the equilibrium point is unique and the convergence of link volumes to the equilibrium state from an arbitrary initial point is guaranteed. Friesz et al. [3] applied tatonnement adjustment processes from classical microeconomic equilibrium models to predict day-to-day traffic system evolution in response to changes in demand. They analyzed the behavior of day-to-day trajectories from disequilibrium under complete or incomplete information provision using the tatonnement mechanism and discussed its stability properties. Nagurney and Zhang [8, 9, 23, 24] introduced the projected dynamical system approach to study the stability of a route choice adjustment process in fixed and elastic demand networks. The route choice model is link-based and the stationary point of this dynamical system coincides with the UE solution for the static traffic assignment problem. Cascetta [2] investigated a stochastic route choice mechanism in which users choose paths according to logit or probit probabilities based on perceived costs that are a moving average of those prevailing on the previous days. A route choice process is considered stable if the process is autonomous: $M$-dependent Markov and irreducible. Cantarella and Cascetta [1] considered the stability of stochastic equilibrium for general networks. They propose regularity conditions to ensure the existence and uniqueness of a stationary probability distribution of system states. Watling [20, 21] extended Horowitz’s results [5] to general networks. A dynamical adjustment process was proposed to study the stability of the general asymmetric stochastic equilibrium assignment problem. The stability analysis is based on the linear approximation of the original non-linear model. It provides a capability to analyze the local stability about the equilibrium point. Peeta and Yang [13, 14, 15] extended Smith’s [18] route switching principle to construct a dynamical systems based model to solve the DTA problem for the system optimal (SO) and UE objective. A general framework was proposed to address the theoretical stability analysis. It can be applied to both static and dynamic traffic assignment problems. This model is a reactive model implying that the model only uses the current traffic information as system inputs. Yang [22] proposed three different control structures to implement Peeta and Yang’s [13, 14, 15] model based on the amount of information available, and developed a deployable solution algorithm that can operate in sub-real time while maintaining a robust level of solution accuracy.

Following the general model proposed by Peeta and Yang [15, 22], this study focuses on the fundamental modeling issues and the insights on system behavior and properties. We first address the key concepts of time-dependent link performance functions and the solution existence. Then, a two-link network is used to demonstrate that the DTA model moves the system towards the time-dependent stable desirable states prescribed by the problem objective.
The Stable Traffic Assignment Model

We first summarize the general DTA model proposed by Peeta and Yang [15, 22]. The model seeks to assign time-dependent paths to vehicles through a stable real-time information provision control strategy that satisfies some system-wide and/or individual user objectives. Its information provision scheme follows a feedback control structure. The traffic control center (as a controller) collects traffic information (as feedback signals) such as instantaneous link travel costs and flows from the field. Based on this information, it provides guidance (as control inputs) to drivers in the traffic network (as a plant) to improve the real-time system performance. The feedback control structure is used to provide real-time route guidance during the duration of interest (congested traffic flow period), typically called the planning horizon. The planning horizon is divided into small consecutive intervals, labeled assignment intervals, for which assignment proportions to the various paths for route guidance are determined. The O-D demand estimates for the next assignment interval are assumed to be available. The real-time measurements of current traffic flows and travel times are used in conjunction with the stable traffic assignment model to obtain a stable solution for the next assignment interval. The variables are defined as follows:

\[ \begin{align*}
N & \quad : \text{the set of nodes} \\
A & \quad : \text{the set of directed arcs} \\
\tau & \quad : \text{the assignment time interval in the traffic system} \\
t & \quad : \text{time unit in the dynamical system} \\
\Omega & \quad : \text{the set of O-D pairs in the network} \\
\omega & \quad : \text{an O-D pair in the network, } \omega \in \Omega \\
\omega \equiv (n, s) & \quad : \text{the O-D pair from node } n \text{ to node } s \\
P & \quad : \text{the set of paths in the network} \\
P_\omega & \quad : \text{the set of paths for O-D pair } \omega, \ P_\omega \subset P \\
p_{i,\omega} & \quad : \text{path } i \text{ for O-D pair } \omega, \ p_{i,\omega} \in P_\omega \\
x_{i,\omega}^m(t) & \quad : \text{flow at the start node } n \text{ of path } i \text{ for O-D pair } \omega \equiv (n, s) \text{ that is considered for assignment by the control strategy in assignment interval } \tau, \text{ at time } t \text{ in the dynamical system} \\
x^m(t) & \quad : \text{the vector of flows } x_{i,\omega}^m(t) \text{ of all paths } i, \ i \in P_\omega, \text{ for O-D pair, } x^m \equiv x^m(t) \\
x^\tau(t) & \quad : \text{the vector of all flows } x^m(t) \text{ for all O-D pairs, } x^\tau \equiv x^\tau(t) \\
x(t) & \quad : \text{the vector of all flows } x^\tau(t) \text{ for all } \tau \\
f^\tau_a(t) & \quad : \text{the traffic flow contribution on link } a \text{ due to the path flows } x^\tau(t) \text{ at time } t \text{ in the dynamical system, } f^\tau_a \equiv f^\tau_a(t) \\
f^\tau(t) & \quad : \text{the vector of all link flows } f^\tau_a(t), \forall a \in A \\
c^\tau_a & \quad : \text{the instantaneous travel cost on link } a \text{ in assignment interval } \tau \\
c^\tau_a(\cdot) & \quad : \text{the instantaneous time-dependent link performance function on link } a \text{ in assignment interval } \tau \\
c^\tau & \quad : \text{the vector of the instantaneous link travel costs on all links}
\end{align*} \]
\( C_i^{\tau o}(x^\tau(t)) \): the instantaneous path travel cost for traffic flow \( x_i^{\tau o}(t) \), \( C_i^{\tau o} \equiv C_i^{\tau o}(x^\tau(t)) \); it is the sum of the corresponding link costs; \( C_i^{\tau o} = \sum_{a \in P_i} c^\tau_a \)

\( C^{\tau o}(x^\tau(t)) \): the vector of instantaneous path travel costs for all paths of the O-D pair \( \omega \) corresponding to the traffic flow \( x^{\tau o}(t) \), \( C^{\tau o} \equiv C^{\tau o}(x^\tau(t)) \)

\( C^{\tau o}(x^\tau(t)) \): the vector of path travel costs \( C^{\tau o} \) for all O-D pairs, \( C^{\tau} \equiv C^{\tau}(x^\tau(t)) \)

\( T_i^{\tau o}(x) \): the instantaneous path travel time for traffic flow \( x_i^{\tau o}(t) \)

\( \hat{T}_j^{\tau o}(x) \): the instantaneous path marginal travel time for traffic flow \( x_i^{\tau o}(t) \)

\( \hat{x}_{j \rightarrow i}^{\tau o}(t) \): the switch rate of the traffic flow \( x_j^{\tau o}(t) \) from path \( j \) to path \( i \) at time \( t \) in the dynamical system, \( \hat{x}_{j \rightarrow i}^{\tau o} \equiv \hat{x}_{j \rightarrow i}^{\tau o}(t) \)

\( \hat{x}_i^{\tau o}(t) \): the net switch rate for path \( i \) of O-D pair \( \omega \) at time \( t \) in the dynamical system, \( \hat{x}_i^{\tau o} \equiv \hat{x}_i^{\tau o}(t) \)

The assignment strategy is based on the following assumption: the switching rate of traffic from path \( j \) to path \( i \) at decision node \( n \) at time \( t \) in the dynamical system is proportional to the path cost difference and the traffic flow \( x_j^{\tau o}(t) \) being considered for assignment. In this context, every decision node \( n \) and destination \( s \) can be treated as an O-D pair. Mathematically, it can be written as:

\[
\hat{x}_{j \rightarrow i}^{\tau o}(t) = \alpha \phi(C_j^{\tau o}, C_i^{\tau o})x_j^{\tau o}(t) ,
\]

where

\[
\phi(y_i, y_k) = \begin{cases} 
  y_i - y_k , & \text{if } y_i - y_k > 0 \\
  0 , & \text{if } y_i - y_k \leq 0 
\end{cases}
\]

and

\[
\alpha > 0.
\]

\( x_j^{\tau o}(t) \) includes two components: (i) new demand originating at node \( n \) initially assigned to path \( j \) for the assignment interval \( \tau \), and (ii) existing flow on the network destined to \( s \) that reaches node \( n \) of path \( j \) in interval \( \tau \). Model (1) describes the magnitude and direction of the flow switching between two paths. For path \( i \), the switching rates from other paths to it are positive and the switching rates from path \( i \) to other paths are negative. The net switching rate for path \( i \) can be obtained by summing all positive and negative switching rates associated with path \( i \), then:

\[
\hat{x}_i^{\tau o}(t) = \sum_{j \in P_i \setminus \{i\}} \alpha \phi(C_j^{\tau o}, C_i^{\tau o})x_j^{\tau o}(t) - \sum_{j \in P_i \setminus \{i\}} \alpha \phi(C_i^{\tau o}, C_j^{\tau o})x_i^{\tau o}(t) .
\]

The first term in equation (2) is the summation of all switch-in rates for path \( i \) and the second term is the summation of the switch-out rates. Model (2) can be re-written in matrix form for an O-D pair \( \omega \) connected by multiple paths:

\[
\hat{x}^{\tau o}(t) = \Phi^{\tau o}(x) \cdot x^{\tau o}(t),
\]

where
Furthermore, we can consider the general network with multiple O-D pairs by putting the models (3) for all O-D pairs together:

$$\dot{x}(t) = \Phi(x) \cdot x(t),$$

where

$$\Phi(x) = \begin{bmatrix}
\Phi^{t_1}(x) & 0 & \cdots & 0 \\
0 & \Phi^{t_2}(x) & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \Phi^{t_{|p|}}(x)
\end{bmatrix}$$

Matrix $$\Phi(x)$$ is a block-diagonal matrix that incorporates the cost difference matrices $$\Phi^{t_{ij}}(x)$$ for each O-D pair.

The desirable state that the system approaches depends on the problem objective. The route switching model (4) will move the system towards different desirable states by using different measures of travel costs in $$\Phi(x)$$ that represent the different problem objectives. Following Wardrop’s [19] first principle, UE is reached only when no driver can improve his/her travel time by unilaterally switching routes. It is characterized by travel times for used paths being equal, and less than or equal to the travel time on unused paths. The UE goal can be addressed by using the instantaneous path travel times $$T^{t_{ij}}(x)$$ to represent the travel costs $$C^{t_{ij}}(x)$$ in model (4). Then, the system follows the UE principle to assign traffic flow $$x^{t_{ij}}$$ in each assignment interval $$\tau$$, and the network approaches dynamic instantaneous UE [13, 14, 15, 22]. By replacing $$C^{t_{ij}}$$ by $$T^{t_{ij}}$$ in model (4), the UE assignment model for general networks can be stated as

$$\dot{x}^{t_{ij}}(t) = \Phi^{t_{ij}}(x) \cdot x^{t_{ij}}(t),$$

where

$$\Phi^{t_{ij}}(x) = \begin{bmatrix}
\Phi^{t_1}_{UE}(x) & 0 & \cdots & 0 \\
0 & \Phi^{t_2}_{UE}(x) & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \Phi^{t_{|p|}}_{UE}(x)
\end{bmatrix}.$$
Following Wardrop’s [19] second principle, the used paths and the associated flows under system optimality are such that the total system travel time is minimized. The SO state is characterized by used paths for an O-D pair having the same marginal travel costs, and the unused paths having equal or higher marginal travel costs. The instantaneous path marginal travel time $\tau_{\omega_i,T}$ is defined as the contribution of an additional user on path $i$ to the total travel time on the network under current traffic conditions. A comprehensive discussion on marginal travel times is provided in Peeta [11, 12]. The instantaneous path marginal travel time is defined as:

$$\tau_{\omega_i,T} = \partial \left( \sum_{a \in A} f_{a_i}(t) \cdot c_{a_i}(f_{a_i}(t)) \right) \partial \left( x_{\omega_i}(t) \right).$$

(6)

The SO assignment model can be formulated by replacing $T_{\omega_i}$ by $\hat{T}_{\omega_i}$ in model (5),

$$\dot{x}^i(t) = \Phi^i_{SO}(x) \cdot x^i(t).$$

(7)

The SO assignment model moves the system towards the corresponding dynamic instantaneous SO state [13, 14, 15, 22].

Time-Dependent Link Performance Functions

A key concept in the proposed models is the use of time-dependent link performance functions in the dynamical system solution procedure. This section discusses the important issues and assumptions associated with these functions. It provides the theoretical foundation for interpreting some ill-behaved phenomena in dynamic traffic networks.

A typical link performance function describes the relationship between link flows and travel costs. The time-dependent link performance function describes the time-dependent relationship between link flows and travel costs as illustrated in Figure 1. It is assumed to be a function of all link flows in the interval $\tau$, enabling the accounting of link interactions. The instantaneous travel cost $c_{a_i}$ is based on two components: (1) the previously assigned flow that is still present in the network and is not considered for assignment or has no opportunity to switch routes in assignment interval $\tau$, termed background flow, and (2) the flow $x^i(t)$ seeking assignment in the interval $\tau$. The background flow is a fixed quantity for assignment interval $\tau$ because the vehicles belonging to background flow do not have an opportunity to

$$\Phi_{UE}^{T} = \left[ \begin{array}{ccc}
- \sum_{a \in P_{\omega_i}[1]} \alpha \phi(T_{\omega_i}, T_{\omega_i}) & \alpha \phi(T_{\omega_2}, T_{\omega_1}) & \ldots & \alpha \phi(T_{\omega_p}, T_{\omega_1}) \\
\alpha \phi(T_{\omega_1}, T_{\omega_2}) & - \sum_{a \in P_{\omega_i}[2]} \alpha \phi(T_{\omega_2}, T_{\omega_1}) & \ldots & \ldots \\
\alpha \phi(T_{\omega_1}, T_{\omega_p}) & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots 
\end{array} \right].$$

(5c)
switch their routes. In other words, \( c_a^\tau(t) \) is a function of only the traffic flows seeking assignment in interval \( \tau \). Therefore, \( c_a^\tau(t) \) can be represented as a function of \( f^\tau(t) \); denoted as \( c_a^\tau(f^\tau(t)) \).

\[
\begin{align*}
\text{Travel cost } c_a^\tau \\
\text{Flow } f_a^\tau \\
\text{Assignment interval } \tau
\end{align*}
\]

Figure 1 Time-Dependent Link Performance Functions for Link \( a \)

\( c_a^\tau(t) \) is also assumed to be positive, continuously differentiable, and monotonically increasing with respect to each \( f_a^\tau \), for all \( a \in A \). It is asymptotic to the link capacity, implying that as the traffic flow approaches capacity a very high penalty is incurred. \( c_a^\tau(t) \) explicitly considers links interactions but does not make limiting assumptions on them, precluding the guarantee of uniqueness for the equilibrium state as can be expected in general traffic networks. The monotonic increasing and continuous differentiability assumptions do not necessarily hold across assignment time intervals. This implies that, for the same link flow, the link travel costs may be different in different assignment intervals. It is consistent with traffic dynamics in time-dependent networks.

Within the same assignment interval \( \tau \), \( c_a^\tau(t) \) follows three assumptions: constancy, differentiability, and monotonic increasing. The constancy assumption states that the shape of the link performance function is fixed within an assignment interval. Since the capacity and the ambient traffic conditions may vary over time due to complexities inherent to dynamic traffic networks, the shape of the link performance function may change with time. To discretize the solution approach, it is assumed that the function changes from one assignment interval to the next, but remains unchanged within an assignment interval. Hence, smaller the assignment interval, it is more likely that variations in traffic conditions are adequately represented, thereby enhancing accuracy. Theoretically, the three assumptions hold true if the assignment interval is sufficiently small. However, as the assignment interval decreases, the computational burden increases. Therefore, it is not realistic to limitlessly reduce the assignment interval length. The constancy assumption for a link performance function within an assignment interval may be violated if the interval length is not sufficiently small and an
incident or dramatic demand changes occur. This can lead to a violation of the continuous differentiability assumption if the functional form changes significantly, and the monotonic increasing assumption as well. A violation of the monotonic increasing assumption implies that as the number of vehicles traveling on a link increase, the travel time for an individual vehicle reduces. This is not a typically scenario in traffic networks, but can occur under some incident situations as shown in Figure 2. It can be conceptually better clarified by assuming that the link performance function changes substantially within the assignment interval itself leading to a seeming violation of the monotonic increasing assumption. However, each of the two performance functions within the assignment interval \( \tau \) satisfy the monotonic increasing assumption as illustrated in Figure 2. This can consistently explain the possibility that travel time reduces even though flow increase after a significant traffic event. Instead, it can be viewed as a violation of the constancy assumption.

![Diagram](image)

**Figure 2 Violations of Link Performance Function Assumptions**

From the perspective of real-time traffic operations, the above modeling approach for \( c_a^\tau(\cdot) \) has several advantages vis-à-vis adequately replicating traffic flow dynamics. The use of the time-dependent link performance functions circumvents the key difficulty of violating the monotonic increasing and/or continuous differentiability assumptions across time intervals, while enabling an adequate representation of ambient traffic conditions. This is especially critical under highly congested situations where traffic conditions in the vicinity of a link can significantly influence its travel costs, thereby implying significant link interactions. A synergistic modeling aspect in this regard is the assumption that the travel time on any link \( a \) depends on the vector of link flows on all links. This enables the accounting of link interactions and the vagaries of dynamic traffic flow phenomena. The potential limitation of assuming a constant monotonically increasing and continuously differentiable link
performance function that is constant within an assignment interval is the possible lack of robustness in accounting for significant traffic events, for example, incidents, which occur during that assignment interval, as illustrated in Figure 2. This suggests smaller assignment intervals to neutralize this issue, leading to the usual trade-offs between computation and accuracy vis-à-vis implementation. An interesting issue vis-à-vis the practical implementation of this modeling approach is the effect that the violation of the monotonic increasing and continuous differentiability assumptions for a few links has on the network itself. If the effects of these violations are insignificant and are suppressed by their satisfaction on all other links, the modeling approach would be adequate for most practical scenarios.

Solution Existence

This section addresses the solution existence of the general traffic assignment model (4).

**Theorem 1. Existence Theorem**

Consider a system \( \dot{x}(t) = F(x) \) with initial point \( x(0) = x_0 \). The continuity of \( F(x) \) guarantees the existence of solutions [4, 6].

**Proposition 1. Solution Existence for Traffic Assignment Model (4)**

**Proof.** Let \( F(x^\tau) = \alpha \cdot \phi(C_i^\tau(x^\tau), C_j^\tau(x^\tau)) \cdot x_j^\tau(t) \). From Theorem 1, if \( F(x^\tau) \) is a continuous function, then the existence of solutions for model (1) is guaranteed. By definition, \( c_a^\tau \) and \( x_i^\tau(t) \) are continuous. Since \( C_i^\tau = \sum_{a=1}^{\tau} c_a^\tau \), \( C_i^\tau \) is also continuous. By definition of the function \( \phi() \), \( \alpha(C_j^\tau - C_i^\tau)x_j^\tau(t) \) is a continuous function, as \( C_j^\tau > C_i^\tau \). Thus, \( F(x^\tau) \) is continuous. When \( C_j^\tau < C_i^\tau \), \( F(x^\tau) = 0 \), which is also continuous. The only possibility for discontinuity is at the point \( C_i^\tau = C_j^\tau \). We prove that the LHS and the RHS limits of \( F(x^\tau) \) exist and are equal when \( C_i^\tau = C_j^\tau \), to conclude that \( F(x^\tau) \) is a continuous function.

Since \( c_a^\tau \) increases monotonically with \( f_a^\tau \ \forall a \in A \), \( C_i^\tau \) and \( C_j^\tau \) also increase monotonically with traffic flows \( x_i^\tau \ \forall i \in P \) because \( C_i^\tau = \sum_{a=1}^{\tau} c_a^\tau \) and \( f_a^\tau(t) = \sum_{a=1}^{\tau} \sum_{i=1}^{\Omega} \delta_{ai}^\tau \cdot x_i^\tau(t) \).

Let \( y = [ y_1^1, y_2^1, ... | y_1^2, y_2^2, ... | y_1^3, y_2^3, ... | y_1^\tau, y_2^\tau, ..., y_j^\tau, y_j^\tau, y_j^\tau, ... ]^T \) be the system state when \( C_i^\tau(y) = C_j^\tau(y) = C \). Here, \( y_k^\tau \geq 0 \ \forall k \in P \). It follows that

\[
F(y^\tau) = \alpha \phi(C_j^\tau(y), C_i^\tau(y)) y_j^\tau(t) = 0,
\]

because \( C_i^\tau(y) = C_j^\tau(y) = C \).
Let \( \widetilde{y} = [y_{1i}^{no}, y_{2i}^{no}, \ldots | y_{1j}^{no}, y_{2j}^{no}, \ldots | y_{1}^{no}, \ldots, y_{i}^{no}, \ldots, \widetilde{y}_{j}^{no}, y_{ji1}^{no}, \ldots]^T \). For any \( \widetilde{y}_{j}^{no} \neq y_{j}^{no} \), it follows that either \( \widetilde{y}_{j}^{no} > y_{j}^{no} \) or \( \widetilde{y}_{j}^{no} < y_{j}^{no} \). If \( \widetilde{y}_{j}^{no} > y_{j}^{no} \), then:

\[
C_{j}^{no}(\widetilde{y}) \rightarrow C, \text{ as } \widetilde{y}_{j}^{no} \rightarrow y_{j}^{no}
\]

\[\Rightarrow [C_{j}^{no}(\widetilde{y}) - C_{i}^{no}(\widetilde{y})] \cdot \widetilde{y}_{j}^{no} \rightarrow 0, \text{ as } \widetilde{y}_{j}^{no} \rightarrow y_{j}^{no}\]

\[\Rightarrow F(y_{j}^{no}) \rightarrow 0, \text{ as } \widetilde{y}_{j}^{no} \rightarrow y_{j}^{no} \]

Hence, the RHS limit \( \lim_{\widetilde{y}_{j}^{no} \rightarrow y_{j}^{no}^+} F(y_{j}^{no}) = 0 \).

A similar proof can be used for the case \( \widetilde{y}_{j}^{no} < y_{j}^{no} \). Therefore, the LHS limit \( \lim_{\widetilde{y}_{j}^{no} \rightarrow y_{j}^{no}^-} F(y_{j}^{no}) = 0 \).

Similarly, for all \( \widetilde{y}_{k}^{no} \neq y_{k}^{no}, k \in P \), it follows that either \( \widetilde{y}_{k}^{no} > y_{k}^{no} \) or \( \widetilde{y}_{k}^{no} < y_{k}^{no} \). Using the same logic, we can show that

\[\lim_{\hat{x}_{i}^{no} \rightarrow x_{i}^{no}^+} F(x_{i}^{no}) = \lim_{\hat{x}_{k}^{no} \rightarrow x_{k}^{no}^-} F(x_{k}^{no}) = 0, \text{ for all } k \in P .\]

Since the LHS and RHS limits are equal, \( F(\cdot) \) is continuous. Hence, \( \alpha phi(C_{j}^{no}, C_{i}^{no})x_{j}^{no}(t) \) is continuous and the existence of solutions for system (1) is guaranteed.

\[\hat{x}_{i}^{no} \text{ is continuous because it is a summation of continuous functions } \alpha phi(C_{j}^{no}, C_{i}^{no})x_{j}^{no}(t) \text{ (see equation 2). From Theorem 1, solutions exist for each } \hat{x}_{i}^{no} \text{ in system (3a). Consequently, the solution existence for the system (4a) is guaranteed.} \]

**System Properties and Characteristics**

The non-negativity and flow conservation constraints are, typically, two essential constraints in traffic assignment problems. The non-negativity constraints state that the traffic flows along paths have physical meaning only when they are non-negative. The flow conservation constraints imply that the total traffic flows for each O-D pair is a constant given a traffic demand level. However, models (4), (5), and (7) do not contain any constraint. An important property of models (4), (5), and (7) is that all feasible traffic flow patterns for a given demand level compose a positively invariant set [15, 22]. A positively invariant set is a set such that if a solution belongs to it at some time \( t \) in the dynamical system, the dynamical system solution belongs to that invariant set for all future time \( t \). This property enables the solution search procedure in the dynamical system to automatically satisfy the two constraints if the system starts at a feasible initial solution. Therefore, models (4), (5), and (7) implicitly satisfy solution non-negativity and flow conservation without explicitly incorporating any constraint.
Models (5) and (7) ensure stability by moving the system towards the equilibrium states prescribed by the corresponding objective rather than seeking the equilibrium states themselves. This is a realistic operational perspective because these states may never be reached on-line given the complexities inherent to dynamic traffic networks. This implies three advantages in the context of real-time operations. First, the search procedure in a dynamical system can be truncated at any time because any intermediate solution is better than all previous intermediate solutions along the search trajectory. This flexibility provides the trade-offs between on-line computational efficiency and solution effectiveness for practical implementation. Second, by focusing on a set of equilibrium states defined by different demands, it enables addressing networks with time-varying demand. Third, the stability analysis provides a global perspective of system behavior unlike some approaches that are restricted to analyzing the local behavior of a specific equilibrium state.

Insights of System Behavior: Two-link Network Example

The aim of this analysis is to demonstrate the ability of model (4) to handle time-dependent demand. Each time-dependent demand matrix corresponds to a particular assignment interval $\tau$ when the control strategy is applied. A simple example is used to provide insights on the system behavior. Consider the traffic network illustrated in Figure 3 that consists of one O-D pair and two links (paths). We apply the UE objective model (5) in this network for different traffic demands. The link performance functions for the two links are assumed as follows:

Path 1: traffic flow $x_1$, performance function $c_1 = 2x_1^2 + 1$, and
Path 2: traffic flow $x_2$, performance function $c_2 = 2x_2$.

![Two-Link Network](Figure 3)

We numerically simulate the system behavior under different demands. Figure 4 illustrates the simulation results in the state portrait diagram. The $x$ axis represents the traffic flow on path 1 and the $y$ axis represents the traffic flow on path 2. The asterisks represent the UE solutions corresponding to the specific network demands, indicated by the straight lines. Hence, the straight line joining (0,6) and (6,0) represents all cases with a demand of 6 units.
Model (5) always moves in the direction of the corresponding UE solution. Therefore, we say that this system is stable because it is bounded and converges to the set of asterisk points. Since the system is directed towards the corresponding equilibrium points defined by different demands, it provides a capability to address networks with time-dependent demand. When the demand changes, the system moves to another straight line, but approaches the corresponding UE state. An important observation from this example is that the system will remain in the first quadrant if the initial state is in the first quadrant (see Figure 4). This behavior corresponds to the non-negativity in path flows implying only non-negative flows have physical meaning. Another observation is that for a given traffic demand \( d \), the system moves only along the diagonal line that satisfies \( x_1 + x_2 = d \). Hence, each diagonal line is a positively invariant set representing a particular demand level. This property corresponds to the flow conservation of the O-D flows.

![Figure 4 State Portrait for The Two-Link Network](image)

Figure 5 plots the Lyapunov function with respect to the state variables. The Lyapunov function used here follows the one proposed by Peeta and Yang [15, 22]. The Lyapunov function for a system is not unique, and it may be possible to construct the function in different ways. For example, Smith’s [18] stability analysis has a quadratic Lyapunov function. Peeta and Yang [15, 22] explored a general framework for theoretical analysis and used the objective functions of UE and SO DTA models as the corresponding Lyapunov functions. Figure 5 provides a visualization of the system behavior. As before, the asterisks represent the equilibrium states defined by the different demands. Each dotted line that intersects an asterisk is a positively invariant set, and the equilibrium state is at the lowest point of the dotted line (which is the asterisk). The dotted lines compose a convex surface like a valley. If the system starts at any point inside a positively invariant set, which is a dotted line in Figure 5, it only moves along that invariant set. Since the Lyapunov function is the corresponding DTA objective function, the bottom of the convex surface is equal to the
minimum DTA solution for that invariant set. In addition, the system starting at any initial point approaches the minimum solution for the corresponding invariant set. Hence, the objective function moves towards the minimum value from any point except at the minimum point itself. This is consistent with the first two conceptual stability definitions stated earlier. Let us suppose that a perturbation disturbs the system residing at the bottom of the surface and moves it away from its equilibrium state. After the perturbation disappears, the system will move back towards the bottom of the surface because its move direction is always towards to the bottom of the convex surface. However, it may not go back to its equilibrium before the perturbation because the perturbation may be caused by additional demand. In this case, the system moves to a different invariant set and will finally approach the equilibrium in that new invariant set. It implies that model (5) always moves the traffic network back to an equilibrium state after a perturbation, which is consistent with the third conceptual stability definition. The two-link network example provides insights on the characteristics of model (5) vis-à-vis stability, as well as the associated system behavior. It also relates the theoretical stability analysis to the conceptual stability definitions.

Figure 5 Visualization of the System Behavior for the Two-Link Example

Conclusions

This study addresses real-time traffic assignment strategies in general traffic networks equipped with advanced information systems. The problem is formulated as a non-linear dynamical system within a feedback control structure. It drives the system to the prescribed objectives based on the current network conditions. The dynamical system based approach is
used to generate the DTA path assignment proportions. The two commonly used assignment criteria, UE and SO, form the objectives for the corresponding assignment strategies analyzed. The traffic assignment strategies can be applied to general networks and provide a stable trajectory for seeking the corresponding objectives. The solution existence is proved to reinforce the theoretical foundation of the model.

The use of time-dependent link performance functions enables studying the time-varying behavior of dynamic traffic networks. This study discusses the assumptions and implications of the proposed time-dependent link performance functions. The consistency of these assumptions with the ill-behaved flow phenomena of dynamic traffic networks is discussed.

A two-link network is used to illustrate the system behavior for the proposed models. The numerical simulation and visualization of the evolution of the Lyapunov functions seamlessly tie the theoretical stability analysis to the conceptual stability definitions. The example provides insights on the system properties, as well as the system behavior in dynamic networks with time-dependent demand.

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