Quiz #1 (ECE3075 Summer 2004): 06/08/04

There are six problems, worth a total of 120 points. You should choose five of them to finish first. Then work towards extra credit by doing the remaining one.

Problem 1 (20%)

The bivariate Gaussian density function of r.v.'s X and Y can be expressed as:

$$f(x,y)=\frac{1}{(\sqrt{2\pi})^2\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\{-\frac{1}{2(1-\rho^2)}[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}-\frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}]\}$$

(a) Find the marginal density function, $f_X(x)$, is X Gaussian?

(b) Find the conditional density function, $f(x\mid y)$, is X given $Y=y$ Gaussian?

(c) What are the conditional mean and conditional variance in (b)?

(d) Compare with conditional variance in (b) and the variance in (a), what can you conclude?

[Hint: Consider first transforming X and Y to standardized random variables. This is Eq. (3-34) you were asked to prove on p. 139 of Cooper and McGillem.]

Problem 2 (20%)

Let X, Y and Z be three standardized (zero mean, unit variance) random variables. X and Y are uncorrelated, X and Z have a correlation coefficient of ½, and Y and Z have a correlation coefficient of -½. Define a new random variable $W=X+Y+Z$.

(a) Find the means and variances of W;

(b) Find the correlation coefficient between W and X;

(c) Let $U=Y-Z$, find the correlation coefficient between W and U.

Problem 3 (20%)

Given two random sinusoidal signals, $X(t)=\cos(100t+\Theta)$ and $Y(t)=\cos(100t+\Psi)$, in which Θ and Ψ are uniformly distributed random phase angles between 0 and 2π, and they are independent of each other. We add the two signals to form a new random signal $Z(t)=X(t)+Y(t)$ [This is Problem 3-5.3 of Cooper and McGillem].

(a) Show that the sum can be expressed as $Z(t)=A\cos(100t+\Phi)$ in which both the amplitude A and the phase Φ are random variables. Express the new pair of random variables A and Φ as functions of Θ and Ψ;

(b) Find the probability density function of Φ [recall Θ and Ψ are independent];

(c) Find the probability that the value of the phase Φ is less than $\pi/3$;

(d) Find the probability that the amplitude A is greater than 1.

Problem 4 (20%)

Given an unbiased deck of 52 cards, each is drawn independently as follows:

(a) Drawing two cards, find the probability of having a blackjack hand;

(b) Continuing from part (a), in a two-person Blackjack game (a dealer and a player), find the probability that both players have blackjack hands;

(c) Drawing five cards, find the probability of having a full house, i.e. three of the five cards forming three-of-a-kind, and the other two forming a pair;

(d) Continuing from part (c), find the probability of a four-of-a-kind hand.

Problem 5 (10%)

A certain typist sometimes makes mistakes by hitting a key to the right or left of the intended key, each with a probability of 0.02. On the standard QWERTY keyboard, the letters E, R, and T are adjacent to one another, and in English these letters occur with probabilities of $P(\text{E})=0.1031$, $P(\text{R})=0.0484$ and $P(\text{T})=0.0796$, respectively (Remember HW#1.1 in counting occurrence of English letters in text, this is Cooper and McGillem, Problem 1-7.2).

(a) What is the probability with which the letter R appear in text typed by this typist? [Hint: Define conditional events needed to compute this probability]

(b) What is the probability with which the letter R appear in text typed by this typist will be in error?

Problem 6 (20%)

Following the definition of an r. v. X having a binomial distribution, i.e. we have the probability of k independent successes in n trials as $P_n(X=k) = {}_nC_k\, p^k (1-p)^{n-k}$, with p being the probability of success, and ${}_nC_k \triangleq n!/[k!*(n-k)!]$.

(a) Find the mean and variance of X based on the definition of the mathematical expectation of discrete random variable $E[g(k)] = \sum_{k=0}^{n} g(k)P_n(k)$ in this case [Hint: ${}_nC_k = (n/k) * {}_{n-1}C_{k-1} = [n(n-1)/k(k-1)]\,{}_{n-2}C_{k-2}, E[X^2] = E[X(X-1)] + E[X]$];

(b) It is known that $E[X^m] = j^{-m} * d^m\phi(u)/du^m \,|_{u=0}$, with the characteristic function of X given as $\phi(u) = (1-p+pe^{ju})^n$ here. Find the mean and variance of X this way. Do your answers agree with what you obtained in part (a) above?

(c) (***Extra Credit*** +10%) Suppose we are trying to predict the result of a national election in a two-person race. With one hundred thousand early votes counted, Candidate Anderson has 49,000 of these votes. Does this imply his opponent will win? [Hint: By assuming $p=0.5$ for not favoring either candidate, then over 99.7% of the time (from the attached table on Standard Normal Curve Areas), the actual votes counted should fall within $\pm 3\sigma$ of the expected number of votes, if we apply a Gaussian approximation to the binomial distribution of X because n is large in this case.]