Solution to Homework 5 (ECE3075 Summer 2004)

1. (10%) Work out Problem 5-1.2 in Cooper and McGillem.

   Y is U(0.000, 0.999) $\Rightarrow \overline{Y} = 0.5, X(t) = \sum_{i=0}^{9} Y(t-i), \overline{X} = \sum_{i=0}^{9} \overline{Y}(t-i)$.

   (a) $X(4.5) = \overline{Y}(0) + \overline{Y}(1) + \overline{Y}(2) + \overline{Y}(3) + \overline{Y}(4) = 2.5;$ (b) $X(9.5) = 5;$ (c) $X(20.5) = 5.$

2. (10%) Work out Problem 5-2.2 in Cooper and McGillem.

   A Gaussian random process *x(t)* (mean=2 and variance=4) is passed through a half-wave rectifier, i.e. output=0 if input<0, and output=input if input>=0.

   a) $$f(x_p) = \{\frac{1}{\sqrt{2\pi} * 2} \exp[-\frac{1}{2*4}(x_p - 2)^2]\} / P(x > 0) \; [(x_p > 0), \text{ with } P(x>0) = Q(1)]$$ $$= \frac{1}{\sqrt{2\pi} * 2Q(1)} \exp[-\frac{1}{2*4}(x_p - 2)^2] = \frac{1}{4.28} \exp[-\frac{1}{8}(x_p - 2)^2] \; (x_p > 0)$$;

   b) $P(x < 0) = 1 - Q(1) \Rightarrow f(x_n) = \{[1 - Q(1)]\delta(x_n)\} / P(x < 0) = \delta(x_n)$;

   c) Let $y = x_p * x_n = 0, \; \forall x \Rightarrow f(y) = \delta(y)$..

3. (10%) Work out Problem 5-3.1 in Cooper and Mcgillem.

   (c) and (f) are deterministic, (a), (b), (d) and (e) are non-deterministic.

4. (10%) Work out Problem 5-3.2 in Cooper and Mcgillem.

   $X(t) = At + B \; (t > 0), A$ is $N(0,9)$, B is $U(0,6)$, A and B are independent.

   (a) $E[X(t)] = E[A]*t + E[B] = 0*t + 3$;

   (b) $E[X^2(t)] = E[A^2]*t^2 + 2t*E[A]E[B] + E[B^2] = 9t^2 + 12 \Rightarrow \text{Var}[X] = 9t^2 + 3$;

   (c) $X(2) = 10, X(4) = 20 \Rightarrow A = 5, B = 0 \Rightarrow X(8) = 40$.

5. (10%) Work out Problem 5-4.1 in Cooper and Mcgillem.

   (b), (d), (e) and (f) are stationary processes. (a) describes traffic flow which is a function of time, (c) is also a time function; so (a) and (c) are non-stationary.

6. (10%) Work out Problem 5-4.2 in Cooper and McGillem.

   $X(t) = A\cos(\omega t + \theta)$, A and ω are constant, and θ is random.

   $$E[X(t)] = 0 \text{ and } E[X(t_1)X(t_2)] = \frac{1}{2\pi}\int_0^{2\pi} A^2 \cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)d\theta$$

   Using $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\frac{\alpha+\beta}{2}) + \cos(\frac{\alpha-\beta}{2})]$ (Appendix A), we have

   a) $$E[X(t_1)X(t_2)] = \frac{A^2}{4\pi}\int_0^{2\pi} \{\cos(\frac{\omega}{2}[t_1 - t_2]) + \cos(\frac{\omega}{2}[t_1 + t_2] + \theta)\} d\theta$$ $$= \frac{A^2}{2}\cos(\frac{\omega\Delta t}{2}) \; (\Delta t = t_1 - t_2 \text{ and the second term is always 0})$$

   Therefore $X(t)$ is a wide-sense stationary process .

b) $X(t)$ is not wide-sense stationary if θ is not U(0, 2kπ) because the second term is not 0 .

7. (10%) Work out Problem 5-5.1 in Cooper and Mcgillem.

$X(t) = A,$ and A is a Rayleigh r. v. with $f(a) = \frac{a}{\sigma^2}\exp[-\frac{a^2}{2\sigma^2}]U(a).$

$E[X(t)] = E[A] = \sigma\sqrt{\pi/2},$ and $\mathrm{Var}[X(t)] = 2\sigma^2 \Rightarrow E[X(t_1)X(t_2)] = (2+\pi/2)\sigma^2$, a constant

a) *X(t)* is wide-sense stationary because the correlation is a constant;

b) $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}X^2(t)dt = A^2 \neq (2+\pi/2)\sigma^2 \Rightarrow X(t)$ is non-ergodic .

8. (10%) Work out Problem 5-5.3 in Cooper and Mcgillem.

$X(t) = \sum_{n=-\infty}^{\infty} Af(t-nT-t_0)$, A and T are constant, and t_0 is $U(0,T)$,

with the function defined as $f(t) = 1, 0 \le t \le T/2$.

For any given t only one k with an interval from 0 to T (near t) is defined for t_0,

a) therefore $E[X(t)] = \frac{A}{T}\int_0^T f(t-t_0)dt_0 = A/2,$;

therefore $E[X(t_1)X(t_2)] = A^2 * E[f(t)] = \frac{A^2}{T}\int_0^T f(t_1-t_0)f(t_2-t_0)dt_0 = A^2/2$

b) $< X(t) >= \frac{1}{T}\int_0^T Af(t-t_0)dt = \frac{A}{2}, < X^2(t) >= \frac{1}{T}\int_0^T A^2 f^2(t-t_0)dt = \frac{A^2}{2}$;

c) Yes, this process can be stationary;

d) Yes, the process can be ergodic.

9. (10%) Work out Problem 5-6.2 in Cooper and Mcgillem.

i	*x*(*i*)	*i*	*x*(*i*)	*i*	*x*(*i*)	*i*	*x*(*i*)	*i*	*x*(*i*)	*i*	*x*(*i*)	*i*	*x*(*i*)
0	0.19	3	0.83	6	-1.47	9	-0.31	12	0.57	15	-0.82	18	0.91
1	0.29	4	-0.01	7	-1.24	10	1.18	13	0.95	16	-0.25	19	-0.19
2	1.44	5	-1.23	8	-1.88	11	1.70	14	1.45	17	0.23	20	0.24

$\Rightarrow n = 21, \bar{x} = 0.0362, \sigma_{\bar{x}}^2 = \sigma_X^2/n = 1/21$(if known). If it has to be estimated, then we can use $\tilde{S}_2^2 = \frac{1}{n-1}[\sum_i x_i^2 - n\bar{x}^2] = 1.051$ (assuming the variance unknown).

10. (20%) Work out Problem 5-6.3 in Cooper and Mcgillem (MATLAB).

Left as an exercise: (a) 9.44; (b) 0.3536; (c) 4.9939 (d) they are close.