Solution to Homework 4 (ECE3075 Summer 2004)

1. (10%) Work out Problem 4-2.2 in Cooper and McGillem (MATLAB).
   Left as an exercise: $\mathrm{E}(\hat{\bar{\beta}}_{\bar{X}}) = 5, \mathrm{Var}(\hat{\bar{\beta}}_{\bar{X}}) = \sigma_{\bar{X}}^2 / n = \sigma_{\bar{X}}^2 / 10 = [100/12]/10 = 0.8333$. Using 30 sets of 10 random samples each, generated from a uniform distributed random variable and repeated five times, we obtain the following corresponding set of empirical variances: 0.6025, 1.0185, 0.8093, 0.9737, and 0.6338.

2. (10%) Work out Problem 4-2.6 in Cooper and McGillem.
   $\beta_A = 120, \beta_N = 120, \sigma_A^2 = 100, \sigma_N^2 = 25, N = 40, P(X \in A) = P(X \in N) = 20/40 = 1/2,$
   $\beta_{\bar{X}} = E[X \mid X \in A]P(X \in A) + E[X \mid X \in N]P(X \in N) = 120/2 + 120/2 = 120,$
   $\sigma_{\bar{X}}^2 = \mathrm{Var}[X \mid X \in A]P(X \in A) + \mathrm{Var}[X \mid X \in N]P(X \in N) = (100 + 25)/2 = 62.5$
   a) Picking 5 transistors with replacement, then $\mathrm{Var}(\hat{\bar{\beta}}_{\bar{X}}) = \sigma_{\bar{X}}^2 / n = \sigma_{\bar{X}}^2 / 5 = 12.5$;
   b) Picking 5 without replacement, then $\mathrm{Var}(\hat{\bar{\beta}}_{\bar{X}}) = (\sigma_{\bar{X}}^2 / n)[(N-n)/(N-1)] = 11.218$;
   c) For s.d. to be 2, we have $\mathrm{Var}(\hat{\bar{\beta}}_{\bar{X}}) = (\sigma_{\bar{X}}^2 / n)[(N-n)/(N-1)] = 4 \Rightarrow n \geq 12$.

3. (10%) Work out Problem 4-3.2 in Cooper and Mcgillem.
   To obtain an unbiased variance estimate, we use the estimate $\tilde{S}_2^2$, with
   $\mathrm{Var}(\tilde{S}_2^2) = n(\mu_4 - \sigma^4)/(n-1)^2$, and set $\mathrm{Var}(\tilde{S}_2^2) \leq (0.02)^2(\sigma^2)^2$. Since
   $\mu_4 =$ fourth central moment$=1 * 3\sigma^4$ [recall HW3-9, and Eq. (2.27)],
   we have $0.00004(n\text{-}1)^2 \geq 2n$, or $n^2 - 5002n + 1 \geq 0 \Rightarrow n \geq 5002$.

4. (10%) Work out Problem 4-3.3 in Cooper and Mcgillem.
   If X is uniformly distributed over 0 and 2π, then $\sigma^2 = [\int_{-\pi}^{\pi} x^2 dx]/(2\pi) = 2\pi^3 / 6\pi = \pi^2 / 3$, and the fourth central moment, $\mu_4 = [\int_{-\pi}^{\pi} x^4 dx]/(2\pi) = \pi^4 / 5$. Since we are interested in an unbiased estimate of the variance, $\mathrm{Var}(\tilde{S}_2^2) = n(\mu_4 - \sigma^4)/(n-1)^2 \leq (0.05)^2(\sigma^2)^2$, or $n(4\pi^4 / 45)/(n-1)^2 \leq (0.05)^2(\pi^2/3)^2 \Rightarrow n^2 - 322n + 1 \geq 0$, or $n \geq 322$.

5. (10%) Work out Problem 4-4.2 in Cooper and McGillem.
   a) $q\% = 90\% \Rightarrow k = 1.645$ (from Appendix E). With $n = 150$ and $\sigma = 10$,
      so $\bar{X} - k\sigma/\sqrt{n} < \hat{\bar{X}} < \bar{X} + k\sigma/\sqrt{n} \Rightarrow 118.657 < \hat{\bar{X}} < 121.343$;
   b) For $n = 21$, $k\sigma/\sqrt{n} = 3.590 \Rightarrow 116.41 < \hat{\bar{X}} < 123.59$;

6. (10%) Work out Problem 4-4.3 in Cooper and Mcgillem.
   With one-sided confidence interval: $\bar{X} - k\sigma/\sqrt{n} < \hat{\bar{X}} < \infty$, with $q\% = 90\% \Rightarrow k = 1.28$.
   a) With $n = 150$: $k\sigma/\sqrt{n} = 1.0451 \Rightarrow$ 90% confidence interval is $118.955 < \hat{\bar{X}} < \infty$.
   b) With $n = 21$: $k\sigma/\sqrt{n} = 2.7932 \Rightarrow$ 90% confidence interval is $117.207 < \hat{\bar{X}} < \infty$.

7. (10%) Work out Problem 4-5.2 in Cooper and Mcgillem.

$n=50,\ \bar{X}=100,\ \sigma=10,\ \bar{x}=115$. Form a two-sided test statistic: $z=\dfrac{\bar{x}-\bar{X}}{\sigma/\sqrt{n}}=15*\sqrt{50}/10=10.61.$

a) At a 95% confidence level, $z_c=1.96<10.61$ (Appendix D) $\Rightarrow$ claim rejected.

b) At a 90% confidence level, $z_c=1.645<10.61$ (Appendix D) $\Rightarrow$ claim rejected.

8. (10%) Work out Problem 4-5.5 in Cooper and Mcgillem.

a) $\bar{X}=100,\ n=9$: we observe $\bar{x}=[\sum x_i]/n=99.78$;

b) $\tilde{\bar{S}}_2^2=[\sum(x_i-\bar{x})^2]/(n-1)=38.94 \Rightarrow \tilde{\bar{S}}_2=6.24$

We form one-sided T-test statistic (8 degrees of freedom):

c) $t=\dfrac{\bar{x}-\bar{X}}{\sigma/\sqrt{n}}=(99.87-100)/6.24\sqrt{9}=-0.107,$

At a 95% confidence level, $t_c=1.86$ (from Table 4-2),

Since $-t_c=-1.86<-0.107 \Rightarrow$ the claim is valid.

9. (10%) Work out Problem 4-6.2 in Cooper and Mcgillem.

a) The scatter plot on a semi-log scale should be easy (left as an exercise);

x	.0001	.001	.01	.1	1	10
$z=\log(x/0.0001)$	1	2	3	4	5	6
y	310	290	285	270	260	225

$n=6, \sum y_i=1640, \sum z_i=21, \sum y_i z_i=5475, \sum z_i^2=91$

b) $\Rightarrow \hat{b}=[6*5475-21*1640]/[6*91-21^2]=-15.14,$

$\hat{a}=[1640-(-15.14)*21]/6=326.33,$

and therefore the linear regression for y on z is $y=326.33-15.14z$.

10.(10%) Work out Problem 4-6.3 in Cooper and Mcgillem (MATLAB).

Left as an exercise using linear regression techniques (hyperbolic fit, with a=1.5660, b=1.1585, resulting in a good curve fit as shown in the figure).